9/2/24.

Q. $\int_{-\pi/2}^{\pi/2} \sin^6 x \, dx.$

Ans. $I = \int_{-\pi/2}^{\pi/2} \sin^6 x \, dx.$

Let $f(x) = \sin^6 x$

$\sin^{2n} x$ is even.

$\therefore I = 2\int_{0}^{\pi/2} \sin^6 x \, dx$

$= \cancel{2}\left[\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{\cancel{2}}\right)\right]$

$= \frac{15\pi}{48} = \frac{5\pi}{16}$

Q. $\int_{-\pi/2}^{\pi/2} \sin^7 x \, dx$

Ans. $I = \int_{-\pi/2}^{\pi/2} \sin^7 x \, dx$

Let $f(x) = \sin^7 x \, dx$

$\sin^{2n+1} x$ is an odd func^n.

$\therefore I = 0$

Q. $\int_{-\pi/2}^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx.$

Ans. $I = \int_{-\pi/2}^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx$

9/2/24

Let $f(x) = \sin^4 x \cdot \cos^6 x$.

$\sin^4 x$ is even

$\cos^6 x$ is even.

$\therefore f(x)$ is an even func^n.

$$\int_{-\pi/2}^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx = 2\int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx$$

$$I = \left[\frac{(3\cdot 1)\cdot(5\cdot 3\cdot 1)}{(10)(8)(6)(4)(2)}\right]\frac{\pi}{\cancel{2}} \times \cancel{2}$$

$$= \frac{9\pi}{480\times 8} = \frac{3\pi}{\cancel{10800}\ 256}.$$

Q. $\int_{-\pi/2}^{\pi/2} \sin^4 x \cdot \cos^7 x \, dx$

Ans. $I = \int_{-\pi/2}^{\pi/2} \sin^4 x \cdot \cos^7 x \, dx$

Let $f(x) = \sin^4 x \cdot \cos^7 x$

$\sin^4 x$ is even

$\cos^7 x$ is even.

(even)(even) = even.

$\therefore f(x)$ is even func^n.

$$I = 2\int_0^{\pi/2} \sin^4 x \cdot \cos^7 x \, dx.$$

$$2\left[\frac{(3\cdot 1)(6\cdot 4\cdot 2)}{\cancel{(10)}(11)(9)(7)(5)(3)(1)}\right](1) = \frac{32\pi}{1155}$$

9/2/24.

Q. ~~$\int_0^{\pi} x \sin$~~

Property:

1. $\int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx$

Q. $\int_0^{\pi} x \sin^5 x \cdot \cos^2 x \; dx.$

Ans. $I = \int_0^{\pi} x \sin^5 x \cdot \cos^2 x \; dx$

$= \int_0^{\pi} (\pi - x) \sin^5(\pi - x) \cos^2(\pi - x) \; dx$

Let $f(\pi - x) = (\pi - x) \sin^5(\pi - x) \cos^2(\pi - x)$

$\pi \sin^5(\pi - x) \cos^2(\pi - x) - x \sin^5(\pi - x) \cos^2(\pi - x)$

$I = \int_0^{\pi} \pi \sin^5(\pi - x) \cos^2(\pi - x) - \int_0^{\pi} x \sin^5(\pi - x) \cos^2(\pi - x)$

$I = \int_0^{\pi} \pi \sin^5(\pi - x) \cos^2(\pi - x) - I$

$2I = \pi \int_0^{\pi} \sin^5(\pi - x) \cos^2(\pi - x) \; dx$

$\frac{2I}{\pi} = \left[\frac{(4 \cdot 2)(1)}{7 \cdot 5 \cdot 3} \right] (1)(2)$

$\left[\because \int_0^{\pi} \sin^m x \cdot \cos^n x \; dx = 2 \int_0^{\pi/2} \sin^m x \cos^n x \; dx, \right.$ if n is even

9/2/24

$$2I = 2\pi\left[\frac{(4\cdot 2)\ (1)}{7\cdot 5\cdot 3}\right](1)$$

$$I = \frac{8\pi}{105}$$

Q. $\int_0^{\pi} x \sin^7 x \cdot \cos^4 x \, dx$

Ans. $I = \int_0^{\pi} x \sin^7 x \cdot \cos^4 x \, dx.$

$$= \int_0^{\pi} (\pi - x)(\sin^7(\pi - x)) \cos^4(\pi - x) \, dx$$

$$= \int_0^{\pi} \pi \sin^7(\pi - x) \cos^4(\pi - x) - \int_0^{\pi} x \sin^7(\pi - x) \cos^4(\pi - x) \, dx$$

$$= \int_0^{\pi} \pi \sin^7 x \cos^4 x \, dx - \int_0^{\pi} x \sin^7 x \cos^4 x \, dx$$

$$2I = \int_0^{\pi} \pi \sin^7 x \cos^4 x \, dx$$

$$\frac{2I}{\pi} = 2\int_0^{\pi/2} \sin^7 x \cos^4 x \, dx$$

$$\frac{I}{\pi} = \left[\frac{(6\cdot 4\cdot 2)\ (3\cdot 1)}{11\cdot 9\cdot 7\cdot 5\cdot 3}\right](1)$$

$$I = \frac{16\pi}{3465}$$

Q. $\int_0^{\pi} x \sin^4 x \cos^4 x \, dx$

Ans. $I = \int_0^{\pi} x \sin^4 x \cos^4 x \, dx.$

$$= \int_0^{\pi} (\pi - x) \sin^4(\pi - x) \cos^4(\pi - x) \, dx.$$

$$= \int_0^{\pi} (\pi - x) \sin^4(x) \cos^4(x) \, dx.$$

$$I = \int_0^{\pi} \pi \sin^4 x \cos^4 x \, dx - \int_0^{\pi} x \sin^4 x \cos^4 x$$

$$I = \int_0^{\pi} \pi \sin^4 x \cos^4 x \, dx - I$$

$$2I = \int_0^{\pi} \pi \sin^4 x \cos^4 x \, dx$$

$$I = \pi \int_0^{\pi/2} \sin^4 x \cos^4 x \, dx$$

$$I = \pi \left[\frac{(3\cdot1)(3\cdot1)}{8\cdot6\cdot4\cdot2} \right] \left(\frac{\pi}{2} \right)$$

$$I = \frac{9\pi^2}{768} = \frac{3\pi^2}{256}$$

Q. $\int_0^{\pi} x \sin^4 x \cos^5 x \, dx.$

Ans. $I = \int_0^{\pi} x \sin^4 x \cos^5 x \, dx$

$$= \int_0^{\pi} (\pi - x)(\sin^4(\pi - x)) \cos^5(\pi - x) \, dx$$

$$I = -\pi \int_0^{\pi} \sin^4 x \cos^5 x \, dx + x \int_0^{\pi} \sin^4 x \cos^5 x \, dx$$

$$I = I - \pi \int_0^{\pi} \sin^4 x \cos^5 x \, dx.$$

$$0 = -\pi \int_0^{\pi} \sin^4 x \cos^5 x \, dx$$

$$= -2\pi \int_0^{\pi/2} \sin^4 x \cos^5 x \, dx.$$

$$= -2\pi \left[\frac{(3\cdot1)(4\cdot2)}{9\cdot7\cdot5\cdot3} \right] (1)$$

$$0 = -\frac{16\pi}{315} = 0$$

Q. $I = \int_0^{\pi} x \cos^6 x \, dx$

Ans. $I = \int_0^{\pi} x \cos^6 x \, dx$

$= \int_0^{\pi} (\pi - x) \cos^6 (\pi - x) \, dx$

$I = \pi \int_0^{\pi} \cos^6 x \, dx - x \int_0^{\pi} \cos^6 x \, dx$

$I = 2\pi \int_0^{\pi/2} \cos^6 x \, dx - I$

$2I = \pi \left[\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right) \right] \left(\frac{\pi}{2}\right)$

$I = \dfrac{5\pi^2}{32}$

10/2/24

~~Q. The fn~~

Gamma funcⁿ / Gamma integral

The funcⁿ of n defined by the integral

$I = \int_0^{\infty} e^{-x} x^{n-1} dx = \Gamma n$ where $n > 0$,

is called the gamma funcⁿ of n

- denoted by Γn

- the above result can be written as

$\Gamma n+1 = \int_0^{\infty} e^{-x} \cdot x^n \, dx$

10/2/24

Q. $\int_0^\infty e^{-x} \cdot x^3 \, dx$

Ans. $\int_0^\infty e^{-x} \cdot x^3 \, dx = \overline{|n+1} = \overline{|3+1} = \overline{|4}$

Q. $\int_0^\infty e^{-x} \cdot x^{1/2} \, dx$

Ans $\int_0^\infty e^{-x} \cdot x^{1/2} \, dx = \overline{|n+1} = \overline{|1/2+1} = \overline{|3/2}$

Properties:

1) $\overline{|1} = 1$
2) $\overline{|n+1} = n\overline{|n}$
3) $\overline{|n+1} = n!$
4) $\overline{|0} = \infty$
5) $\overline{|1/2} = \sqrt{\pi}$

Type I: reduction of integral in the form of gamma fu

Q. Evaluate $\int_0^\infty \cancel{x} \sqrt{y}\, e^{-\sqrt{y}} \, dy$

Ans. We know that

$\overline{|n+1} = \int_0^\infty e^{-x} \cdot x^n \, dx$

$I = \int_0^\infty y^{1/2} e^{-y^{1/2}} \, dy$

Let $\sqrt{y} = x = y^{1/2}$

$I = \int_0^\infty x e^{-x} \, dy$

$\sqrt{y} = x$

$d\,\cancel{\sqrt{y} = d}$

$y = x^2$

$dy = 2x\,dx$

10/2/24

$$I = \int_0^{\infty} e^{-x} \cdot 2x^2 \, dx$$

$$= 2\int_0^{\infty} e^{-x} x^2 \, dx$$

$2\Gamma 3 = 2! \times 2 = 2 \times 2 = 4$

Q. $\int 4\sqrt{y} \cdot e^{-\sqrt{y}} \, dy$

Ans. We know that,

$$\Gamma n+1 = \int_0^{\infty} e^{-x} \cdot x^n \, dx$$

$$I = \int_0^{\infty} 4 y^{1/2} \cdot e^{-y^{1/2}} \, dy$$

Let $\sqrt{y} = x$

$y = x^2$

$dy = 2x \, dx$

$$I = \int_0^{\infty} 4x \cdot e^{-x} \cdot 2x \, dx$$

$$= 8\int_0^{\infty} x^2 \cdot e^{-x} \, dx$$

$= 8\Gamma 2+1 = 8 \cdot 2! = 16$

Q. $\int_0^{\infty} \sqrt{y} \, e^{-y^3} \, dy$

Ans. We know that,

$$\Gamma n+1 = \int_0^{\infty} e^{-x} \cdot x^n \, dx$$

$$I = \int_0^{\infty} e^{-y^3} \cdot \sqrt{y} \, dy$$

$y^3 = x$

~~$3y^2 dy = dx$~~

10/2/24

$$y^3 = x$$
$$y = x^{1/3}$$
$$dy = \frac{1}{3} x^{-2/3} dx$$

$$I = \int_0^\infty e^{-x} \cdot x^{+1/6} \left(\frac{1}{3}\right) x^{-2/3} dx$$
$$= \int_0^\infty e^{-x} \frac{x^{-1/2}}{3} dx$$
$$= \frac{1}{3} \int_0^\infty e^{-x} \cdot x^{-1/2} dx$$
$$\cancel{= \frac{1}{3} \Gamma\left(-\frac{5}{6}+1\right)} = \frac{1}{3} \Gamma\left(\frac{1}{6}\right)$$
$$= \frac{1}{3} \Gamma\left(-\frac{1}{2}+1\right) = \frac{1}{3} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}$$

Q. $\int_0^\infty y^7 e^{-2y^2} dy$

Ans. We know that,
$$\Gamma(n+1) = \int_0^\infty e^{-x} \cdot x^n dx.$$

$$I = \int_0^\infty y^7 e^{-2y^2} dy$$

$$2y^2 = x$$
$$y^2 = \frac{x}{2}.$$
$$y = \sqrt{\frac{x}{2}} = \frac{x^{1/2 - 2/3}}{\sqrt{2}}$$
$$\cancel{dy = \frac{1}{\sqrt{2}\sqrt{x}} x^{-1/2} = \frac{1}{\sqrt{2x}} dx} \quad dy = \frac{1}{2\sqrt{2}} x^{-1/2} = \frac{1}{2\sqrt{2}\sqrt{x}}$$

10/2/24

$$I = \int_0^\infty e^{-x} \cdot \left(\frac{x^{1/2}}{\sqrt{2}}\right)^7 \left(\frac{1}{\sqrt{2}\,x\cdot 2}\right) dx$$

$$= \frac{1}{2\sqrt{2}} \int_0^\infty e^{-x} \cdot \frac{1}{\sqrt{x}}\, dx$$

$$= \frac{1}{2\sqrt{2}} \int_0^\infty e^{-x} \cdot x^{-1/2}\, dx$$

$$= \left(\frac{1}{2\sqrt{2}}\right) \Gamma\left(-\tfrac{1}{2}+1\right)$$

$$= \Gamma\left(\tfrac{1}{2}\right)\left(\frac{1}{2\sqrt{2}}\right) = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

$$= \int_0^\infty e^{-x} \cdot \left(\frac{x^{7/2-1/2}}{2^{7/2}} \cdot \frac{1}{\cancel{2}\cdot 2^{3/2}\, x}\right) dx$$

$$= \frac{1}{2^5} \int_0^\infty e^{-x}\, x^{-6/2}\, dx$$

$$= 2^{-5} \int_0^\infty e^{-x}\, x^{-3}\, dx$$

$$= \Gamma(+3+1)(2^{-5}) = 2^{-5}\,\Gamma 4 = 2^{-5}\cdot 6 = \frac{6}{32} = 0.1875$$

T

Type II: ~~exp~~ integration $\int_0^\infty \frac{x^a}{a^x}\, dx$.

Q. $I = \int_0^\infty \frac{x^a}{a^x}\, dx$

Ans $I = \int_0^\infty \frac{x^a}{a^x}\, dx$

$$= \int_0^\infty a^{-x} \cdot x^a\, dx$$

10/2/24

$$a^{-x} = e^{-y}$$

$$-x \log a = -y \log e$$

$$x \log a = y \log e$$

$$x = \frac{y \log e}{\log a}$$

$$= \frac{y(1)}{\log a}$$

$$dx = \frac{dy}{\log a}$$

$$\cancel{I = \int_0^\infty \left(\frac{y}{\log a}\right.}$$

$$I = \int_0^\infty e^{-y} \cdot \left(\frac{y}{\log a}\right)^a \left(\frac{dy}{\log a}\right)$$

$$= \int_0^\infty e^{-y} \cdot \frac{y^a}{(\log a)^a (\log a)} \, dy$$

$$= \int_0^\infty e^{-y} \cdot \frac{y^a}{(\log a)^{a+1}} \, dy$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-y} \cdot y^a \, dy.$$

$$= \frac{1}{(\log a)^{a+1}} \cdot \Gamma(a+1)$$

13/2/24

Type III: form $\int_0^\infty a^{bx^n}\,dx$.

Q. Evaluate integral $\int_0^\infty 5^{-4x^2}\,dx$

Ans. Let $I = \int_0^\infty 5^{-4x^2}\,dx$

Let $e^{-y} = 5^{-4x^2}$

$\Rightarrow e^{y} = 5^{4x^2}$

Taking log on both sides.

$4x^2 \log 5 = y \log e$

$4x^2 \log 5 = y$

$x^2 = \dfrac{y}{4\log 5}$

$x = \dfrac{y^{1/2}}{2\sqrt{\log 5}}$

$dx = \dfrac{1}{2\sqrt{\log 5}}\left(\dfrac{1}{2}\right) y^{-1/2}\,dy$

$dx = \dfrac{y^{-1/2}}{4\sqrt{\log 5}}\,dy$

$$I = \int_0^\infty e^{-y}\left(\frac{y^{-1/2}}{4\sqrt{\log 5}}\right)dy$$

$$= \frac{1}{4\sqrt{\log 5}}\int_0^\infty e^{-y}\,y^{-1/2}\,dy$$

$$= \frac{1}{4\sqrt{\log 5}}\,\Gamma\left(-\tfrac{1}{2}+1\right) = \frac{\sqrt{\pi}}{4\sqrt{\log 5}}$$

$$= \frac{1}{4}\sqrt{\frac{\pi}{\log 5}}$$

13/2/24

Q. $\int_0^{\infty} \frac{1}{3^{4x^2}} dx$

Ans. $I = \int_0^{\infty} 3^{-4x^2} dx$

Let $e^{-y} = 3^{-4x^2}$

$e^{y} = 3^{4x^2}$

$y \log e = 4x^2 \log 3$

$y = 4x^2 \log 3$

~~$dy = (4\log 3)(2x)$~~

~~$dy = 8x \log 3$~~

~~$x = dy$~~

$x^2 = \frac{y}{4 \log 3}$

$x = \frac{y^{1/2}}{2\sqrt{\log 3}}$

$dx = \frac{(y^{-1/2})}{2\sqrt{\log 3}} \left(\frac{1}{2}\right) dy$

$dx = \frac{1}{4\sqrt{\log 3}} y^{-1/2} dy$

$I = \int_0^{\infty} e^{-y} \left(\frac{1}{4\sqrt{\log 3}}\right) y^{-1/2} dy$

13/2/24.

$$I = \frac{1}{4\sqrt{\log 3}} \int_0^\infty e^{-y} y^{-1/2}\, dy$$

$$= \frac{1}{4\sqrt{\log 3}} \Gamma\left(-\tfrac{1}{2}+1\right)$$

$$= \frac{1}{4}\sqrt{\frac{\pi}{\log 3}}$$

H.W. Q. $\int_0^\infty \frac{1}{4^{5x^2}}\, dx$

Q. $\int_0^\infty \frac{1}{2^{3x^2}}\, dx$

Type IV: logarithmic functions.

> NOTE:- example of $\log x$ we put $\log x = -y$
> for $\log\left(\frac{1}{x}\right)$ we put $\log \frac{1}{x} = y$

Q. $\int_0^1 x^m (\log x)^n\, dx$

Ans. $I = \int_0^1 x^m (\log x)^n\, dx$

Let $\log x = -y$

$x = e^{-y}$

$dx = -e^{-y}\, dy$

x	0	1
y	∞	0

13/2/24

$$I = \int_0^1 (e^{-y})^m (-y)^n (-e)^{-y} dy$$

$$= -\int_0^1 (e^{-ym})(y)^n (-1)^n (e)^{-y} dy$$

$$= -\int_\infty^0 (e^{-ym})(y)^n (-1)^n (e)^{-y} dy$$

$$= (-1)^n \int_0^\infty e^{-(m+1)y} y^n dy$$

$$= (-1)^n \Gamma(n+1) \qquad \left[\int_0^\infty e^{-kx} x^n dx = \frac{\Gamma(n+1)}{k^{n+1}}\right]$$

$$= (-1)^n \frac{\Gamma(n+1)}{(m+1)^{n+1}}$$

Q. $\int_0^1 (x \log x)^4 dx$

Ans. $I = \int_0^1 (x \log x)^4 dx$

Let $\log x = -y$

$x = e^{-y}$

$dx = -e^{-y} dy$

x	0	1
y	∞	0

$$\cancel{I = \int_\infty^0 (e^{-y})(-y)(e^{-y}) dy}$$

$$\cancel{= -\int_0^\infty e^{-2y}(-y) dy}$$

$$= \int_0^\infty$$

13/2/24

$$I = \int_{\infty}^{0} (e^{-y})^{4} \cdot (-y)^{4}\, dx$$

$$= \int_{\infty}^{0} e^{-4y} (-y^{4}) \cancel{\;} (-e^{-y}) \cdot dy$$

$$= -\int_{0}^{\infty} e^{-4y} (+y)^{4} (e)^{-y} (-1)^{-y}\, dy$$

$$= \cancel{(-1)^{-y}} \int_{0}^{\infty} e^{-5(y)} (y)^{4}\, dy$$

$$= \cancel{(-1)}$$

$$= \frac{\Gamma 4+1}{5^{5}} = \frac{\Gamma 5}{5^{5}} = \frac{4!}{5^{5}} = 7.68 \times 10^{-3}$$

Q. $I = \int_{0}^{1} \frac{x\, dx}{\sqrt{\log \frac{1}{x}}}$

Ans. $I = \int_{0}^{1} \frac{x}{\sqrt{\log \frac{1}{x}}}\, dx$

~~Put~~ -

Let $\log\left(\frac{1}{x}\right) = y$

$-\log x = y$

$x = +e^{y}$

$dx = -e^{-y}\, dy$

x	0	1
y	∞	0

$$I = \int_{\infty}^{0} \frac{e^{-y} (e^{-y})}{\sqrt{y}}\, dy$$

:)

$$I = -\int_{\infty}^{0} e^{-y} e^{-y} y^{-1/2} \, dy$$

$$= \int_{0}^{\infty} e^{-2y} y^{-1/2} \cdot dy$$

$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{2^{\left(-\frac{1}{2}\dots\right.}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}}$$

Beta func^n

The func^n of m and n defined by integral

$$\int_0^1 x^{m-1} (1-x)^{n-1} \, dx = 0. \quad (m>0,\ n>0)$$

~~is calle~~

- denoted $\beta(m,n)$.

Properties:

1. $$\int_0^1 x^m (1-x)^n \, dx = \beta(m+1, n+1)$$

2. $\beta(m,n) = \beta(n,m)$

3. $$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

4. $$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

5. $$\Gamma p \, \Gamma 1-p = \frac{\pi}{\sin p\pi} \quad [0<p<1]$$

Q. $\int_0^{\pi/2} s$

22/2/24

Unit III: Multiple Integration & Its Application

Double Integration

* Case I: $\iint f(x,y)\,dx\,dy = \int_{y_1=a}^{y_2=b}\left[\int_{x_1=\phi(y)}^{x_2=\psi(y)} f(x,y)\,dx\right]dy$

- first integrate w/r to x, then y.

* Case II: $\iint f(x,y)\,dx\,dy = \int_{c}^{d}\left[\int_{\phi(x)}^{\psi(x)} f(x,y)\,dy\right]dx$

- first integrate w/r to y, then x.

Case III: $\iint f(x,y)\,dx\,dy = \int_{a}^{b}\left[\int_{a}^{d} f(x,y)\,dx\right]dy$

- if dx is first then $x = c, d$ otherwise $y = c, d$.

Case IV: if $f(x,y) = h(x)\cdot g(y)$, then

$$\iint f(x,y)\,dx\,dy = \int_c^d\int_a^b h(x)\cdot g(y)\,dx\,dy.$$

$$= \left[\int_{y=c}^{d} g(y)\,dy\right]\left[\int_{x=a}^{b} h(x)\,dx\right]$$

22/2/24.

Q. Evaluate.

$$\int_0^1 \int_0^{1-x} (x+y)\, dx\, dy$$

Ans. $\int_0^1 \left[\int_0^{1-x} (x+y)\, dy \right] dx =$

$$\int_0^1 \left[xy + \frac{y^2}{2} \right]_0^{1-x} dx = \int_0^1 \frac{2x(1-x) + (1-x)^2}{2}\, dx$$

$$= \frac{1}{2} \int_0^1 2x(1-x) + (1-x)^2\, dx$$

$$= \frac{1}{2} \int_0^1 \cancel{2x} - 2x^2 + 1 + x^2 \cancel{-2x}\, dx$$

$$= \frac{1}{2} \int_0^1 1 - x^2\, dx$$

$$= \frac{1}{2} \left[x - \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} \left(1 - \frac{1}{3} \right)$$

$$= \frac{1}{2} \left(\frac{2}{3} \right) = \frac{1}{3}$$

Q. Evaluate :

$$\int_0^1 \int_{2x^2}^{2\sqrt{x}} xy^2\, dx\, dy$$

Ans. $\int_0^1 \left[\int_{2x^2}^{2\sqrt{x}} xy^2\, dy \right] dx =$

$$\left[\int_{2x^2}^{2\sqrt{x}} y^2\, dy \right] \left[\int_0^1 x\, dx \right] =$$

$$\left[\frac{y^3}{3} \right]_{2x^2}^{2\sqrt{x}} \left[\frac{x^2}{2} \right]_0^1 =$$

$$\left[\frac{8x^{3/2}}{3} - \frac{8x^6}{3}\right]\left[\frac{1}{2}\right] =$$

$$\left(x^{3/2} - x^6\right)\left(\frac{8}{6}\right) =$$

$$\frac{4}{3}\left(x^{3/2} - x^6\right)$$

$$I = \int_0^1 \int_{2x^2}^{2\sqrt{x}} xy^2 \, dx \, dy$$

$$= \int_0^1 \left[\int_{2x^2}^{2\sqrt{x}} xy^2 \, dy\right] dx$$

$$= \int_0^1 \left[\frac{xy^3}{3}\right]_{2x^2}^{2\sqrt{x}} dx$$

$$= \int_0^1 \left[\frac{x(8x^{3/2})}{3} - \frac{x(8x^6)}{3}\right] dx$$

$$= \int_0^1 \left[\frac{8x^{5/2}}{3} - \frac{8x^7}{3}\right] dx$$

$$= \frac{8}{3} \left[\frac{2x^{7/2}}{7} - \frac{x^8}{8}\right]_0^1$$

$$= \frac{8}{3}\left[\frac{2(1)}{7} - \frac{1}{8}\right]$$

$$= \frac{8}{3}\left(\frac{2}{7} - \frac{1}{8}\right)$$

$$= \frac{8}{3}\left(\frac{9}{56}\right) = \frac{3}{7} = 0.43$$

22/2/24

Q. $\int_0^1 \int_0^{\sqrt{1-y^2}} xy^2 \, dx \, dy$.

Ans. $I = \int_0^1 \int_0^{\sqrt{1-y^2}} xy^2 \, dx \, dy$

$= \int_0^1 \left[\int_0^{\sqrt{1-y^2}} xy^2 \, dx \right] dy$

$= \int_0^1 \left[\frac{x^2 y^2}{2} \right]_0^{\sqrt{1-y^2}} dy$

$= \int_0^1 \left[\frac{(1-y^2)y^2}{2} \right] dy$

$= \frac{1}{2} \int_0^1 (y^2 - y^4) \, dy$

$= \frac{1}{2} \left[\frac{y^3}{3} - \frac{y^5}{5} \right]_0^1$

$= \frac{1}{2} \left[\frac{1}{3} - \frac{1}{5} \right]$

$= \frac{1}{2} \left(\frac{2}{15} \right) = \frac{1}{15} = 0.67$

Q. ~~[illegible]~~.

Q. $\int_0^1 \int_0^x e^{x+y} \, dx \, dy$

Ans. $I = \int_0^1 \int_0^x e^{x+y} \, dx \, dy$

$= \int_0^1 \left[\int_0^x e^{x+y} \, dy \right] dx$

$= \int_0^1 \left[\int_0^x e^{x} \cdot e^y \, dy \right] dx$

$= \int_0^1 \left[e^x \left[e^y \right]_0^x \right] dx = \int_0^1 e^x (e^x - 1) \, dx$

$$= \int_0^1 e^{2x} - e^x \, dx$$

$$= \left[\frac{e^{2x}}{2} - e^x\right]_0^1$$

$$= \left(\frac{e^2}{2} - e\right) - \left(\frac{e^0}{2} - 1\right)$$

$$= \frac{e^2}{2} - e - \frac{1}{2} + \frac{2}{2}$$

$$= \frac{e^2}{2} - e + \frac{1}{2}$$

$$= \frac{1}{2}\left(e^2 - 2e + 1\right)$$

$$= \frac{1}{2}(e-1)^2$$

13/2/24

Q $\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Ans $I = \int$

Q $\int_0^{\infty} \frac{dx}{1+x^4}$

put $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

x	0	∞
y	0	$\pi/2$

$$I = \int_0^{\pi/2} \frac{\sec^2\theta \, d\theta}{1+\tan^2\theta}$$

$x^2 = \tan\theta$

$x = \tan^{1/2}\theta$

$dx = \frac{1}{2} \tan^{-1/2}\theta \cdot \sec^2\theta \, d\theta$

$$I = \int_0^{\pi/2} \frac{\frac{1}{2}\tan^{-1/2}\theta \cdot \sec^2\theta \, d\theta}{1+\tan^2\theta}$$

$$= \int_0^{\pi/2} \frac{\frac{1}{2}\tan^{-1/2}\theta \, \sec^2\theta}{\sec^2\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\tan^{-1/2}\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \tan^{-1/2}\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$$

$\therefore p = \frac{-1}{2}, \; q = \frac{1}{2}$

$$I = \frac{1}{2}\left[\frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{\frac{1}{2}+1}{2}\right)\right]$$

$$= \frac{1}{4}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{4}\frac{\Gamma\frac{1}{4} \cdot \Gamma\frac{3}{4}}{\Gamma 1}$$

$$= \frac{1}{4}\Gamma\frac{1}{4} \cdot \Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{4}\left[\frac{\pi}{\sin\frac{1}{4}\pi}\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{1/\sqrt{2}}\right]$$

$$= \frac{\pi}{2\sqrt{2}}$$

15/2/24.

Q. Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \cdot \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$.

Ans. $I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \cdot \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

$I = I_1 \cdot I_2$ where $I_1 = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

$I_2 = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

$$I_1 = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$$

$$= \int_0^{\pi/2} \tan^{1/2}\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$$

$p = 1/2$, $q = -1/2$

$$I_1 = \frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2}\Gamma\tfrac{3}{4}\,\Gamma\tfrac{1}{4}$$

$$= \frac{1}{2}\Gamma\tfrac{1}{4}\,\Gamma\left(1-\tfrac{1}{4}\right)$$

$$= \frac{1}{2}\left[\frac{\pi}{\sin \frac{1}{4}\pi}\right]$$

$$I_1 = \frac{1}{2}\,\frac{\pi}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}\,\pi}{2} = \frac{\pi}{\sqrt{2}}$$

$$I_2 = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$$

$$= \int_0^{\pi/2} \cot^{1/2}\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^{1/2}\theta\, d\theta$$

$$= \frac{1}{2} \Gamma\frac{3}{4}\, \Gamma\frac{1}{4}$$

$$= \frac{1}{2}\left[\frac{\pi}{\sin \frac{1}{4}\pi}\right]$$

$$= \left(\frac{1}{2}\right)\frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$I_2 = \frac{\pi}{\sqrt{2}}$$

$$\therefore I = I_1 \cdot I_2$$

$$= \frac{\pi^2}{2}$$

Q. $\int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta \cdot \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$

Ans. $I = \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta \cdot \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$

$I = I_1 \cdot I_2$ where $I_1 = \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta$

$I_2 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$

$$I_1 = \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta$$

$p = -1/2$; $q = 0$

$$I_1 = \frac{1}{2} \frac{\Gamma\frac{1}{4} \cdot \Gamma\frac{1}{2}}{\Gamma\frac{3}{4}}$$

$$= \frac{1}{2} \frac{\Gamma\frac{1}{4} \sqrt{\pi}}{\Gamma\frac{3}{4}}.$$

$$I = I_1 \cdot I_2$$

$$= \left(\frac{\frac{1}{2}\Gamma\frac{1}{4} \cdot \sqrt{\pi}}{\Gamma\frac{3}{4}}\right)\left(\frac{\frac{1}{2}\Gamma\frac{3}{4}\sqrt{\pi}}{\Gamma\frac{5}{4}}\right)$$

$$= \frac{\frac{1}{4}\pi\Gamma\frac{1}{4}}{\Gamma\frac{5}{4}} = \pi$$

Case ~~Type~~ IV: $\Gamma n = (n-1)\Gamma n-1$

$$\int_a^b (x-a)^m (b-x)^n \, dx.,$$

put $x-a = (b-a)y$

1 Q. $\int_2^5 \sqrt{(x-2)^7(5-x)^9}\, dx$

Ans. $I = \int_2^5 \sqrt{(x-2)^7(5-x)^9}\, dx$

$$I = \int_2^5 (x-2)^{7/2} (5-x)^{9/2} \, dx$$

put $x-2 = (5-2)y$

$= 3y$

$x = 3y+2$

$dx = 3 \cdot dy$

x	2	5
y	0	1

$$I = \int_0^1 (3y)^{7/2} (5-2-3y)^{9/2}\, 3\,dy$$

$$= 3\int_0^1 3^{7/2} \cdot y^{7/2} (3-3y)^{9/2}\, dy$$

$$= 3\int_0^1 3^{7/2} y^{7/2}\, 3^{9/2} (1-y)^{9/2}\, dy$$

$$= 3^9 \int_0^1 y^{7/2} (1-y)^{9/2}\, dy$$

$$= 3^9\, \beta\left(7/2+1,\ 9/2+1\right)$$

$$= 3^9\, \beta(9/2,\ 11/2)$$

$$= 3^9 \left(\frac{\Gamma 9/2 \, , \, \Gamma 11/2}{\Gamma 10}\right)$$

$$= 3^9 \left[\frac{\left(\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\right)\cdot\left(\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\right)}{9!}\right]$$

$$= \frac{3^9}{9!}\left[\frac{9\cdot 7^2\cdot 5^2\cdot 3^2(\pi)}{2^9}\right]$$

$$= \cancel{8.51}\ 10.51\,\pi$$

Q. $I = \int_3^7 (x-3)^{1/4} (7-x)^{1/4}\, dx$

Ans. $I = I_1 \cdot I_2$ where $I_1 = \int_3^7 (x-3)^{1/4}$

$$I_2 = \int_3^7 (7-x)^{1/4}\, dx$$

put $x - 3 = (7-3)y$

$$x - 3 = 4y$$

$$x = 4y+3$$

$$dx = 4\,dy$$

x	3	7
y	0	1

15/2/24

$$I = \frac{8\,\Gamma\frac{5}{4}\cdot\Gamma\frac{5}{4}}{\Gamma\frac{5}{2}}$$

$$= \frac{8\left(\frac{1}{4}\Gamma\frac{1}{4}\right)^2}{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}}$$

$$I = \frac{8\cdot\frac{1}{4}\left(\Gamma\frac{1}{4}\right)^2}{3\sqrt{\pi}}$$

$$= \frac{2}{3\sqrt{\pi}}\left(\Gamma\frac{1}{4}\right)^2$$

16/2/24

Q. $\int_2^3 (x-2)^{1/2}(3-x)^{1/3}\,dx.$

Ans. $I = \int_2^3 (x-2)^{1/2}(3-x)^{1/3}\,dx$

$I = I_1 \cdot I_2$

$I_1 = \int_2^3 (x-2)^{1/2}\,dx$ $\quad I_2 = \int_2^3 (3-x)^{1/3}\,dx$

Let $(x-2) = (3-2)y$

$x-2 = y$

$x = y+2$

$dx = 1\,dy$

x	2	3
y	0	1

$$I = \int_0^1 y^{1/2} (3-2-y)^{1/3}\, dy.$$

$$= \int_0^1 y^{1/2} (1-y)^{1/3}\, dy$$

$$= \beta\left(\frac{1}{2}+1, \frac{1}{3}+1\right)$$

$$= \beta\left(\frac{3}{2}, \frac{4}{3}\right)$$

$$= \frac{\Gamma 3/2 \; \Gamma 4/3}{\Gamma 17/6}$$

$$= \frac{\left(\frac{1}{2}\Gamma\frac{1}{2}\right)\left(\frac{1}{3}\Gamma\frac{1}{3}\right)}{\Gamma 17/6}$$

$$= \frac{\frac{1}{6}\sqrt{\pi}\,\Gamma\frac{1}{3}}{\frac{11}{6}\cdot\frac{5}{6}\,\Gamma\frac{5}{6}} = \frac{6}{55}\sqrt{\pi}\,\frac{\Gamma\frac{1}{3}}{\Gamma\frac{5}{6}}$$

Q. $I = \int_0^1 x^2 (1-\sqrt[3]{x})^3\, dx$

Ans. $I = I_1 \cdot I_2$ where $I_1 = \int_0^1 x^2 \, dx$

$I_2 = \int_0^1 (1-\sqrt[3]{x})^3\, dx$

Let $\sqrt[3]{x} = y$

$x = y^3$

$dx = 3y^2\, dy$

x	0	1
y	0	1

$$I = \int_0^1 (y^3)^2 (1-y)^3 \cdot 3y^2\, dy$$

$$= 3\int_0^1 3y^8 (1-y)^3\, dy$$

$$= 3\,\beta(8+1, 3+1)$$

$$= 3\,\beta(9, 4)$$

$$= 3\left(\frac{\Gamma 9 \, \Gamma 4}{\Gamma 13}\right)$$

$$= \frac{3 \times 8! \times 3!}{12!}$$

$$= \frac{3 \times 6}{12 \times 11 \times 10 \times 9_3 \ _2} = \frac{1}{660}$$

Q. $\int_0^\infty e^{-x^2} x^{1/5} \, dx$

Ans. $I = \int_0^\infty e^{-x^2} x^{1/5} \, dx$ - where $I_1 = \int_0^\infty e^{-x^2} dx$

$I_2 = \int_0^\infty x^{1/5} dx$

Let $x^2 = y$

$x = \sqrt{y} = y^{1/2}$

$dx = \frac{1}{2} y^{-1/2} dy$

x	0	∞
y	0	∞

$$I = \int_0^\infty e^{-y} \cdot y^{1/10} \cdot \frac{1}{2} y^{-1/2} dy$$

$$= \frac{1}{2} \int_0^\infty e^{-y} \cdot y^{-4/10} dy$$

$$= \frac{1}{2} \frac{\Gamma -1 \, \Gamma \frac{-4}{10}}{\Gamma \frac{-14}{10}} = \frac{1}{2} \frac{\Gamma -1 \, \Gamma \frac{-2}{5}}{\Gamma \frac{-3}{5}} = \frac{1}{2} \Gamma \frac{-3}{5}$$

Q. $I = \int_0^\infty \frac{x^7}{7^x}\, dx.$

Ans $I = I_1 \cdot I_2$ where $I_1 = \int_0^\infty x^7\, dx$

$I_2 = \int_0^\infty \frac{1}{7^x}\, dx$

Let $7^{-x} = e^{-y}$

$7^x = e^y$

$x \log 7 = y \log e$

$x = \frac{1}{\log 7} y$

x	0	∞
y	0	∞

$$I = \int_0^\infty \left(\frac{1 \cdot y}{\log 7}\right)^7 \cdot e^{-y} \cdot \frac{1}{\log 7}\, dy$$

$$= \left(\frac{1}{\log 7}\right)^8 \int_0^\infty e^{-y} y^7\, dy$$

$$= \left(\frac{1}{\log 7}\right)^8 \Gamma(7+1)$$

$$= \frac{7!}{8 \log 7}$$

Q. $\int_0^{\pi/6} \cos^4(3x)\, dx.$

Ans. $I = \int_0^{\pi/6} \cos^4(3x)\, dx$

~~$I = I_1 + I_2$ where $I_1 = \int_0^{\pi/6}$~~

Let $3x = \theta$

$x = \frac{\theta}{3}$

$dx = \frac{d\theta}{3}$

x	0	$\pi/6$
3	1	

Page

Q. $\int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx$.

Ans. $I = \int_0^{\pi} \sin^4 x \cdot \cos^6 x \, dx$

$= \left[\frac{(3\cdot 1)\cdot(5\cdot 3\cdot 1)}{(10)(8)(6)(4)(2)(1)} \right] = \frac{45}{3840} = 0.012$

Q. $L\frac{dI}{dt} + RI = E$ $R = 120\,\Omega$

$L = 0.6$ henry

$E = 30\ V$

Ans. $\frac{dI}{dt} + \left(\frac{R}{L}\right)I = \frac{E}{L}$ which is L.D.E. in I,

$I.F. = e^{\int \frac{R}{L} dt}$

$= e^{\frac{Rt}{L}}$

G.S.,

$$Ie^{\frac{Rt}{L}} = \int \frac{E}{L} e^{\frac{Rt}{L}} dt + A$$

$$= \frac{E}{L} \cdot \frac{e^{\frac{Rt}{L}}}{\frac{R}{L}} + A$$

$$Ie^{\frac{Rt}{L}} = \frac{E}{R} e^{\frac{Rt}{L}} + A$$

$$I = \frac{E}{R} + Ae^{\frac{-Rt}{L}}$$

At $t=0$, $I=0$.

$$0 = \frac{E}{R} + A$$

$$A = -\frac{E}{R}$$

$$I = \frac{E}{R} - \frac{E}{R} e^{\frac{-Rt}{L}}$$

$$= \frac{E}{R}\left(1 - e^{\frac{-Rt}{L}}\right)$$

$$= \frac{30}{120}\left(1 - e^{\frac{-120}{0.6}t}\right)$$

$$= \frac{1}{4}\left(1 - e^{-200t}\right)$$

UNIT III :

Differentiation under Integral Sign (DUIS)

Rule → Integral with constant limit

If $I(k) = \int_a^b f(x,\alpha)\,dx$ where a & b are constant

then, $\frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)\,dx$.

Q. Prove that ~~Gio~~ $\int_0^1 \frac{x^a - 1}{\log x}\,dx = \log(a+1)$, $a \geq 0$.

Ans. ~~Let I(a) =~~

Ans. Let $I(a) = \int_0^1 \frac{x^a - 1}{\log x}\,dx$

By DUIS Rule,

$$\frac{dI}{da} = \int_0^1 \frac{\partial}{\partial a}\left[\frac{x^a - 1}{\log x}\right] dx$$

$$\cancel{= \frac{1}{\log x}\int_0^1 (x^a \log x)(x^a - 1)\,dx}$$

$$\cancel{= \frac{1}{\log x}\left[x^a \log x - x\right]_0^1}$$

$$\cancel{= \frac{1}{\log x}\left[(1^a \log 1 - 1) - (0)\right]}$$

$$\cancel{\frac{dI}{da} = \frac{1}{\log x}}$$

$$= \int_0^1 \frac{x^a \log x}{\log x} \cdot dx$$

$$= \int_0^1 x^a\,dx$$

$$= \left[\frac{x^{a+1}}{a+1}\right]_0^1$$

$$\frac{dI}{da} = \frac{1^{a+1}}{a+1} - 0 = \frac{1}{a+1}$$

21/2/24.

$$\int dI = \int \frac{1}{a+1}\, da$$

$$I = \log(a+1) + c$$

$$I(a) = \log(a+1) + c$$

Now to find value of c,

NOTE:- either try making log(a+1) term zero or substitute the smallest possible value.

$$I(0) = \log(0+1) + c$$

$$I(0) = 0 + c$$

$$I(0) = c$$

$$I(a) = \int_0^1 \frac{x^a - 1}{\log x}\, dx$$

$$I(0) = \int_0^1 \frac{(0)^a - 1}{\log(0)}\, dx$$

$$= \frac{-1}{\infty}$$

$$\approx 0$$

$$\therefore I(0) = 0$$

$$\Rightarrow c = 0$$

$$\therefore \int_0^1 \frac{x^a - 1}{\log x}\, dx = \log(a+1) \; ; \; a \geq 0.$$

Q. Prove that $\int_0^1 \frac{x^a - x^b}{\log x} dx = \log\left(\frac{a+1}{b+1}\right) \quad a, b \geq 0$

Ans. Let $I(a) = \int_0^1 \frac{x^a - x^b}{\log x} dx = \log\left(\frac{a+1}{b+1}\right) \quad a, b \geq 0$

By DUIS Rule,

$$\frac{dI}{da} = \int_0^1 \left[\frac{\partial}{\partial a}\left(\frac{x^a - x^b}{\log x}\right)\right] dx$$

$$= \int_0^1 \frac{x^a \log x}{\log x} dx$$

$$= \int_0^1 x^a \, dx$$

$$= \left[\frac{x^{a+1}}{a+1}\right]_0^1$$

$$\frac{dI}{da} = \frac{1}{a+1}$$

$$\int dI = \int \frac{1}{a+1} da$$

$$I(a) = \log(a+1) + c \quad \text{—— (2)}$$

Put $a = b$,

From (2),

$$I(b) = \log(b+1) + c$$

From (1),

$$I(b) = \int_0^1 \frac{x^b - x^b}{\log x} dx$$

$$I(b) = \int_0^1 0 \, dx = \log(b+1) + c = 0$$

$$\log(b+1) + c = 0$$

$$c = -\log(b+1)$$

21/2/24

$$I(a) = \log(a+1) + c$$
$$= \log(a+1) - \log(b+1)$$
$$= \log\left(\frac{a+1}{b+1}\right) = 0$$

Q Show that:

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log\left(\frac{b}{a}\right) \quad a, b > 0.$$

Ans. $I(a) = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log\left(\frac{b}{a}\right)$

Taking b as constant.

$I(a) = \int_0^\infty$

$$\frac{dI}{da} = \int_0^\infty \left[\frac{\partial}{\partial a}\left(\frac{e^{-ax} - e^{-bx}}{x}\right)\right] dx$$

$$= \int_0^\infty \left[\left(\frac{e^{-ax}(-x) - 0}{x}\right)\right] dx$$

$$= \int_0^\infty \frac{-\cancel{x}\, e^{-ax}}{\cancel{x}}\, dx$$

$$= \int_0^\infty -e^{-ax}\, dx$$

$$= \left[\frac{+e^{-ax}}{+a}\right]_0^\infty$$

$$= \frac{1}{e^{\infty} a} = 0 - \frac{1}{a}$$

$$\frac{dI}{da} = \frac{-1}{a}$$

$$\int dI = -\int \frac{1}{a}\, da$$

$$I(a) = -\log a + c$$

Let $a = b$

$$I(b) = -\log b + c$$

$$I(b) = \int_0^\infty \frac{e^{-bx} - e^{-bx}}{b}\, dx$$

$$I(b) = \int_0^\infty 0\, dx = 0 = -\log b + c$$

$$\therefore c = \log b.$$

$$I(a) = -\log a + \log b$$

$$= \log b - \log a$$

$$I(a) = \log\left(\frac{b}{a}\right)$$

$$\boxed{\therefore \int_0^\infty \frac{e^{-ax} - e^{-bx}}{b}\, dx = \log\left(\frac{b}{a}\right)}$$

Q. $\displaystyle\int_0^\infty \frac{1-\cos(ax)}{x^2}\, dx.$

Ans. $\displaystyle I(a) = \int_0^\infty \frac{1-\cos(ax)}{x^2}\, dx.$

$$\frac{dI}{da} = \int_0^\infty \left[\frac{\partial}{\partial a}\left(\frac{1-\cos(ax)}{x^2}\right)\right] dx.$$

$$= \int_0^\infty \left[\frac{1}{x^2}\left(\sin(ax)\cdot x\right)\right] dx$$

$$= \int_0^\infty \frac{\sin(ax)}{x}\, dx$$

$$= -\cos(ax)$$

$\cancel{\int ab = a\int b}$.

$$\int dI = \int \frac{\pi}{2}\, da$$

$$I(a) = \frac{\pi a}{2} + c$$

Let $a=0$,

$I(a) = 0 + c$

$I(a) = c$

$I(0) = 0 = c$

$$I(a) = \frac{\pi a}{2}$$

H.W

25/2/24

Maths Tutorial

Q. $\int_0^{\pi/6} \cos^5(3x)\,dx$

Ans. $\int_0^{\pi/6} \cos^5(3x)\,dx = I$

Let $3x = \theta$

$x = \frac{\theta}{3}$

$dx = \frac{d\theta}{3}$

x	0	$\pi/6$
θ	0	$\pi/2$

$I = \int_0^{\pi/2} \cos^5(\theta)\,\frac{d\theta}{3}$

$= \frac{1}{3}\int_0^{\pi/2} \cos^5\theta\,d\theta$

$= \frac{1}{3}\left[\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)\left(\frac{1}{1}\right)\right]\cdot 1$

$= \frac{8}{45}$

Q. $\int_0^{\pi/8} \sin^6(4x)\,dx$

Ans. $I = \int_0^{\pi/8} \sin^6(4x)\,dx$

$4x = \theta$

$x = \frac{\theta}{4}$

$dx = \frac{d\theta}{4}$

x	0	$\pi/8$
θ	0	$\pi/2$

$I = \frac{1}{4}\int_0^{\pi/2} \sin^6(\theta)\,d\theta$

$= \frac{1}{4}\left[\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\right]\frac{\pi}{2}$

$= \frac{15\pi}{384}$

Q. $\int_{-\pi/2}^{\pi/2} \sin^6 x\,dx$

$I = \int_{-\pi/2}^{\pi/2} \sin^6 x\,dx$

$= \int^{\pi/2}$

$I = 2\int_0^{\pi/2} \sin^6 x\,dx$

$= 2\left[\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\right]\cdot\frac{\pi}{2}$

$= \frac{15\pi}{48}$

Q.

23/2/24

Q. $\int_{-\pi}^{\pi} \sin^4 x \cdot \cos^6 x \, dx$.

Ans. $I = 4\int_{0}^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx$

$$I = 4\left[\frac{(3)(1)\cdot(5)(3)(1)}{(10)(8)(6)(4)(2)}\right]\left(\frac{\pi}{2}\right)$$

$$= \frac{45\pi}{1920}$$

Q. $\int_{0}^{\pi} x\cos^4 x \, dx$

Ans. $I = \int_{0}^{\pi} x \cos^4 x \, dx$

$$= \pi \int_{0}^{\pi/2} \cos^4 x \, dx$$

$$= \pi\left[\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\right]\frac{\pi}{2}$$

$$= \frac{3\pi^2}{16}$$

Q: $\int_{0}^{\pi} x \sin^5 x \cos^4 x \, dx$.

Let $x = (\pi - x)$

$$\int_{0}^{\pi} (\pi - x) \sin^5(\pi - x) \cos^4(\pi - x) \, dx =$$

$$\int_{0}^{\pi} (\pi - x) (\sin^5 x)(-\cos x)^4 \, dx$$

$$\int_{0}^{\pi} \pi \sin^5 x \cos^4 x \, dx - \int_{0}^{\pi} x \sin^5 x \cos^4 x \, dx$$

$$I = \int_{0}^{\pi} \pi \sin^5 x \cos^4 x \, dx - I$$

23/2/24

$$\int_0^{\pi} (\pi - x)(\sin^5 x)\cdot(\cos^4 x)\, dx =$$

$$\int_0^{\pi} \pi \sin^5 x \cos^4 x\, dx - \int_0^{\pi} x \sin^5 x \cos^4 x \, dx = I$$

$$I = \int_0^{\pi} \pi \sin^5 x \cos^4 x \, dx + I$$

$$\frac{2I}{\pi} = 2\int_0^{\pi} \sin^5 x \cos^4 x \, dx$$

$$\frac{I}{\pi} = \int_0^{\pi/2} \sin^5 x \cos^4 x \, dx.$$

$$= \left[\frac{(4)(2)\cdot(3)(1)}{(9)(7)(5)(3)(1)}\right](1)$$

$$I = \frac{24\pi}{945}.$$

27/2/24

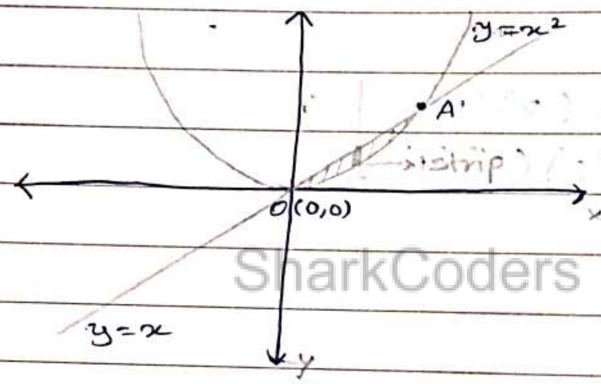
Q. Type II: Evaluation of Double Integration without limit.

Q. Evaluate: $\iint_R (x+y)xy \, dx \, dy$ over the area bounded by $y = x^2$ and $y = x$.

Ans. Let $I = \iint_R xy(x+y) \, dx \, dy$.

where $R: y = x^2$ and $y = x$

$y = x^2$



SharkCoders

Consider strip ‖ y-axis (integrate first w/r to y)

To find A,

$y = x^2$ $y = x$ $\Rightarrow x^2 = x$ $x^2 - x = 0$ $x(x-1) = 0$ $\Rightarrow x = 0$ $x = 1$	$y = (1)^2 = 1$ $y = (1) = 1$ $y = (0)^2 = 0$ $y = 0 = 0$ $\therefore O(0,0)$ and $A(1,1)$

Consider strip ∥ y-axis (integrate first w/r to y)

limit (y) = x^2 (lower end) limit(x) = 0 (lower end)

x (upper end) 1 (upper end)

$$I = \int_{x=0}^{x=1}\left[\int_{y=x^2}^{y=x} xy(x+y)\,dy\right]dx$$

$$= \int_{x=0}^{1}\left[\int_{x^2}^{x} x^2y + xy^2\,dy\right]dx$$

$$= \int_0^1\left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1\left[\left(\frac{x^6}{2} + \frac{x^7}{3}\right) - \left(\frac{x^4}{2} + \frac{x^4}{3}\right)\right]dx$$

$$= \int_0^1\left(\frac{x^6 - x^4}{2} + \frac{x^7 - x^4}{3}\right)dx$$

$$\cancel{= \int_0^1 \left(\frac{3x^6 - 3x^4 + 2x^7 - 2x^4}{6}\right)dx}$$

$$\cancel{= \frac{1}{6}\int_0^1 3x^6 + 2x^7 - 5x^4}$$

$$= \int_0^1\left[\left(\frac{x^4}{2} + \frac{x^4}{3}\right) - \left(\frac{x^6}{2} + \frac{x^7}{3}\right)\right]dx$$

$$= \int_0^1\left(\frac{x^4 - x^6}{2} + \frac{x^4 - x^7}{3}\right)dx$$

$$= \frac{1}{6}\int_0^1\left(3x^4 - 3x^6 + 2x^4 - 2x^7\right)dx$$

$$= \frac{1}{6}\int_0^1\left(5x^4 - 3x^6 - 2x^7\right)dx$$

$$= \frac{1}{6}\left[\frac{5x^5}{5} - \frac{3x^7}{7} - \frac{2x^8}{8}\right]_0^1$$

$$= \frac{1}{6}\left[x^5 - \frac{3x^7}{7} - \frac{x^8}{4}\right]_0^1$$

$$= \frac{1}{6}\left[\left(1 - \frac{3}{7} - \frac{1}{4}\right) - 0\right] = \frac{1}{6}\left(\frac{9}{28}\right) = \frac{9}{168} = \frac{3}{56}$$

22/5/24

Q $\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)(1-y^2)}}$

Ans. $I = \int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$

$\left[\int_0^1 \frac{1}{\sqrt{1-x^2}} dx\right]\left[\int_0^1 \frac{1}{\sqrt{1-y^2}} dy\right] =$

$[\sin^{-1}x]_0^1 \; [\sin^{-1}y]_0^1 =$

$[\sin^{-1}(1) - \sin^{-1}(0)][\sin^{-1}(1) - \sin^{-1}(0)] =$

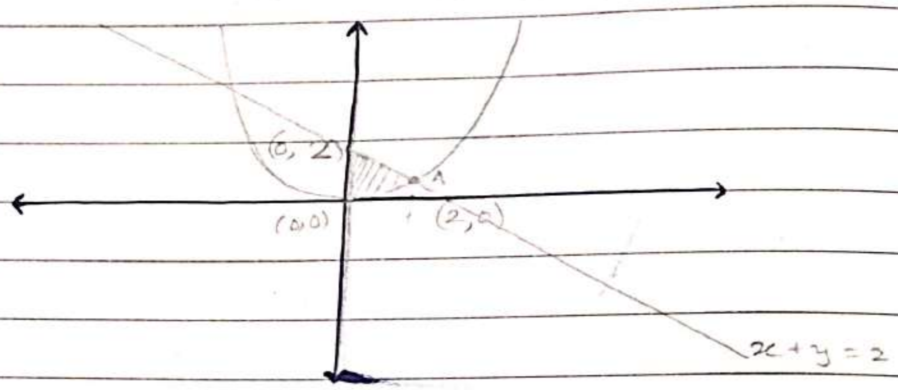
$\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$

27/2/24

Q. Evaluate:

$\iint_R y \, dx \, dy$ over the area bounded by

$y = x^2$, $x = 0$ and $x + y = 2$



To find point of intersection.

$y = x^2$ $\therefore (A) \in (1,1)$

$-y = 2 - x$

$x^2 = 2 - x$

$x^2 + \cancel{x} = 2$

$x^2 + x - 2 = 0$

$x = 1, -2$

$y = 1, x = 1$

Consider strip $\parallel$ y-axis,

limit $(y) = x^2$, $y = 2 - x$

$x = 0$, $x = 1$

$$I = \int_0^1 \left[\int_{x^2}^{2-x} y \, dy \right] dx$$

$$= \int_0^1 \left[\frac{y^2}{2} \right]_{x^2}^{2-x} dx$$

$$= \int_0^1 \left[\cancel{\frac{x^2}{2}} - \frac{x^4}{2} \right] dx$$

27/2/24

$$\cancel{= \frac{1}{2}\int_0^1 (x^2 - x^4)\,dx}$$

$$\cancel{= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1}$$

$$\cancel{= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)}$$

$$\cancel{= \frac{1}{15}}$$

$$I = \int_0^1 \left[\frac{(2-x)^2}{2} - \frac{x^4}{2}\right] dx$$

$$= \frac{1}{2}\int_0^1 (4 + x^2 - 4x - x^4)\,dx$$

$$\cancel{= \frac{1}{2}\int_0^1 4x}$$

$$= \frac{1}{2}\left[4x + \frac{x^3}{3} - \frac{4x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{2}\left[4x + \frac{x^3}{3} - 2x^2 - \frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{2}\left(4 + \frac{1}{3} - 2 - \frac{1}{5}\right)$$

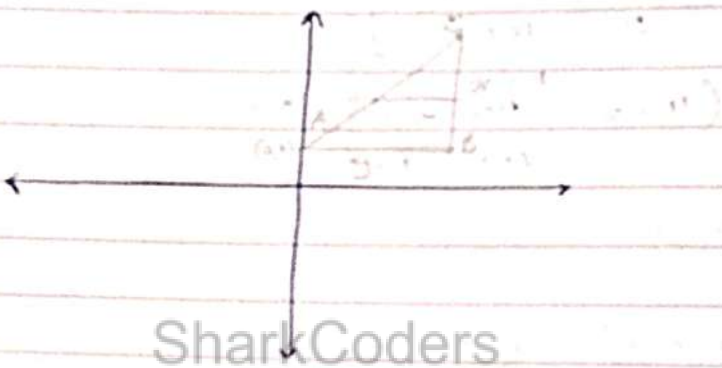
$$= \frac{16}{15}$$

27/8/24

Q.

Ans Let $I = \iint_R (x^2+y^2)\,dx\,dy$ where

R : area of Δ whose vertices are (0,1), (1,1) and (1,



For AC $\Rightarrow \dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$

$$\frac{y-1}{2-1} = \frac{x-0}{1-0}$$

$$y-1 = x$$

$$x = y-1$$

Consider || to x-axis (integrate first w/r to x)

limit (x) = $y-1$ $\quad x = 1$

$y = 1$ $\quad y = 2$

$$I = \int_1^2 \left[\int_{y-1}^1 (x^2+y^2)\,dx \right] dy$$

$$I = \int_1^2 \left[\frac{x^3}{3} + y^2 x \right]_{y-1}^{1} dy$$

$$= \int_1^2 \left[\left(\frac{1}{3} + y^2 \right) - \left(\frac{(y-1)^3}{3} + y^2(y-1) \right) \right] dy$$

$$= \int_1^2 \left[\frac{1}{3} + y^2 - \frac{(y-1)^3}{3} - y^3 + y^2 \right] dy$$

$$= \int_1^2 \left[\frac{1}{3} + 2y^2 - y^3 - \frac{(y-1)^3}{3} \right] dy$$

$$= \int_1^2 \left[\frac{1}{3} + 2y^2 - y^3 - \frac{y^3 + 3y^2 - 3y - 1}{3} \right] dy$$

$$= \frac{1}{3} \int_1^2 \left[\cancel{1} + 6y^2 - 3y^3 - y^3 + 3y^2 - 3y \cancel{-1} \right] dy$$

$$= \frac{1}{3} \int_1^2 \left(9y^2 - 4y^3 - 3y \right) dy$$

$$= \frac{1}{3} \left[\frac{9y^3}{3} - \frac{4y^4}{4} - \frac{3y^2}{2} \right]_1^2$$

$$= \frac{1}{3} \left[3y^3 - y^4 - \frac{3y^2}{2} \right]_1^2$$

$$= \frac{1}{3} \left[\left(24 - 16 - 6 \right) - \left(3 - 1 - \frac{3}{2} \right) \right]$$

$$= \frac{1}{3} \times \frac{3}{2} = \frac{1}{2}.$$

28/2/24

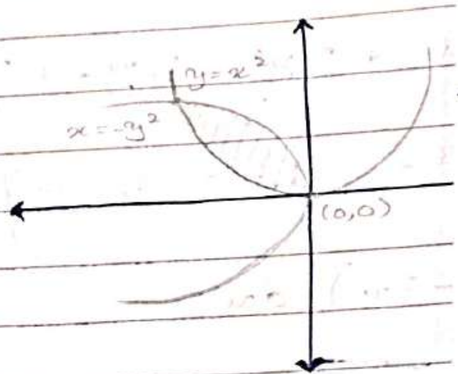
Q. $\iint_R xy \, dx \, dy$ over th

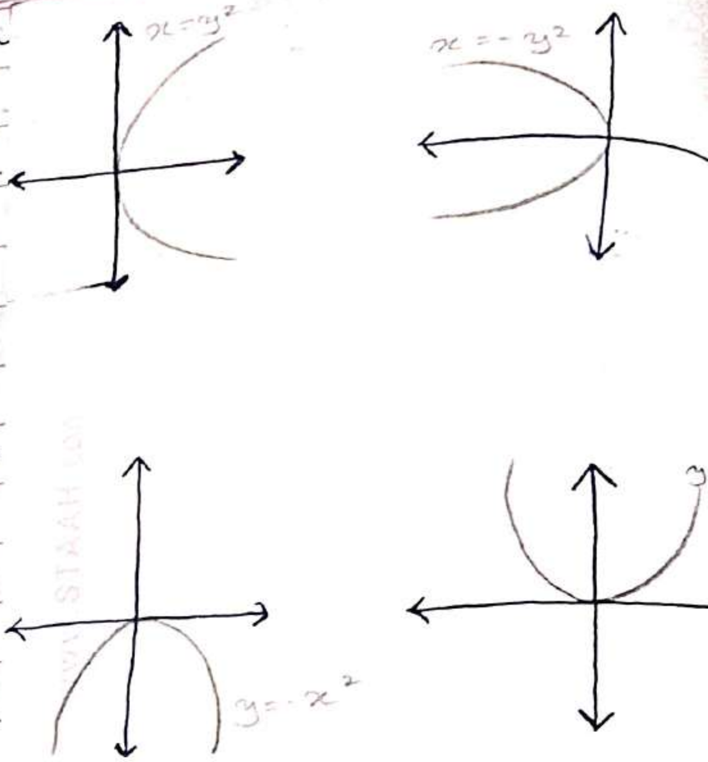
$x^2 = y$ and $y^2 = -x$.

Ans. $y = -x$

Let $I = \iint_R xy \, dx \, dy$

$R: y = x^2$ and $x = -y^2$





To find point of intersection,

$y = x^2$ and $x = -y^2$

$y = (-y^2)^2$ limit

$y = y^4$

$y^4 - y = 0$

$y(y^3 - 1) = 0$

$y = 0$; $y = 1$

$x = 0$; $x = -1$

$O(0,0)$; $A(-1,1)$

Consider $\parallel$ to x-axis (integrate first with x)

$x^2 = y$ $\qquad$ $x = -y^2$

$x = \sqrt{y}$

$y = 0$ $\qquad$ $y = 1$

28/2/24

$$\int_{y=0}^{1}\left[\int_{x=\sqrt{y}}^{y^2} xy\,dx\right]dy = \int_{y=0}^{1}\left[\frac{x^2y}{2}\right]_{\sqrt{y}}^{y^2} dy$$

$$= \int_{y=0}^{1}\left(\frac{y^5}{2}-\frac{y^2}{2}\right)dy$$

$$= \frac{1}{2}\int_{y=0}^{1}(y^5-y^2)\,dy$$

$$= \frac{1}{2}\left[\frac{y^6}{6}-\frac{y^3}{3}\right]_0^1$$

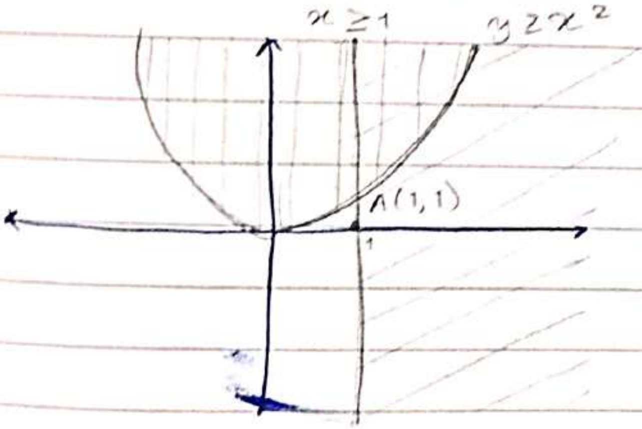
$$= \frac{1}{2}\left(\frac{1}{6}-\frac{1}{3}\right) = \frac{1}{2}\left(\frac{-1}{6}\right)$$

$$= \frac{-1}{12}.$$

Q. Evaluate $\iint_R \frac{dx\,dy}{x^4+y^2}$ over $x \geq 1$ & $y \geq x^2$

Ans. $I = \iint_R \frac{dx\,dy}{x^4+y^2}$ where

$R:\ x \geq 1\ ;\ y \geq x^2.$



To find point of intersection,

$x = 1$ and $y = x^2$

$\Rightarrow x = 1$ and $y = 1$

$\therefore A(1,1)$

Consider ‖ to y-axis (integrate first with y)

limit $(y) = x^2 \qquad y = \infty$

$x = 1 \qquad x = \infty$

$$I = \int_1^\infty \left[\int_{x^2}^\infty \frac{1}{x^4+y^2}\, dy\right] dx$$

$$= \int_1^\infty \left[\int_{x^2}^\infty \frac{1}{(x^2)^2+y^2}\, dy\right] dx$$

$$= \int_1^\infty \left[\frac{1}{x^2} \tan^{-1}\left(\frac{y}{x^2}\right) dy\right] dx$$

$$= \int_1^\infty \frac{1}{x^2} \left[\tan^{-1}(\infty) - \tan^{-1}(1)\right] dx$$

$$= \int_1^\infty \frac{1}{x^2} \left(\frac{\pi}{2} - \frac{\pi}{4}\right) dx$$

$$= \frac{\pi}{4} \int_{x=1}^\infty \frac{1}{x^2}\, dx$$

$$= \frac{\pi}{4} \left[\frac{-1}{x}\right]_1^\infty$$

$$= \frac{\pi}{4}(0+1) = \frac{\pi}{4}$$

Change Double Integration in polar co-ordinate system

Sometimes evaluation of double integration becomes simply by transforming the curve from cartesian form to polar form.

Let $p(x,y) = \varphi(r,\theta)$ with the polar co-ordinates

The polar curve generally is given in the form $r = f(\theta)$.

Put $x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$\therefore x^2 + y^2 = r^2$

and

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \cancel{\theta}$

$\therefore \iint_R f(x,y)\,dx\,dy = \iint_R f(r,\theta)|J|\,dr\,d\theta$ [J = jacobian]

where $|J| \neq 0$,

hence, $|J| = r$

$\therefore \iint_R f(x,y)\,dx\,dy = \iint f(r,\theta)\,r.\,dr\,d\theta$

[$dx\,dy = r\,dr\,d\theta$ where r is jacobian]

H.W.

$x = r\cos\theta \qquad y = r\sin\theta$

$|J| = \frac{\partial(x,y)}{\partial(r,\theta)} = r$

28/2/24.

Type III: Limits are provided.

Q. Evaluate: $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-x^2-y^2}\,dx\,dy$

Ans $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-x^2-y^2}\,dx\,dy.$

NOTE:- Use polar for circle or ellipse

$y = 0$

$y = \sqrt{a^2-x^2}$

$y^2 = a^2 - x^2.$

$x^2 + y^2 = a^2$ ——— (1)

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} e^{-x^2-y^2}\,dx\,dy$$

Given curve is $y = 0$ and $y = \sqrt{a^2-x^2}$
and $x = 0$ and $x = a$

Now we know that

$x^2 + y^2 = a^2$ [From (1)]

... is a circle with ~~ex~~ centre at (0,0) and radius of a.

Put $x = r\cos\theta$ and
$y = r\sin\theta$

$\Rightarrow dx\,dy = r\,dr\,d\theta$

limit $(r) = 0 \qquad r = a$

$(\theta) = 0 \qquad \theta = 2\pi$

28/2/24

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} e^{-r^2}\, r\, dr\, d\theta$$

Let $-r^2 = -t$

$r^2 = t$

$2r\, dr = dt$

$r\, dr = \frac{dt}{2}$

r	0	a
t	0	a^2

$$I = \int_{\theta=0}^{2\pi}\left[\int_{t=0}^{a^2} e^{-t}\,\frac{dt}{2}\right] d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{2}\left[\int_{t=0}^{a^2} e^{-t}\, dt\right] d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{2}\left[\frac{e^{-t}}{-1}\right]_0^{a^2} d\theta$$

$$\cancel{= \int_{\theta=0}^{2\pi} \frac{1}{2}\left(-e^{-a^2}\cdot e + e\right) d\theta} \quad ??$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{2}\left(-e^{-a^2}+1\right) d\theta = \int_{\theta=0}^{2\pi} \frac{1}{2}\left[-e^{-a^2}+1\right] d\theta$$

$$= \frac{1}{2}\left(1-e^{-a^2}\right)\int_0^{2\pi} d\theta$$

$$= \frac{1}{2}\left(1-e^{-a^2}\right)\left[\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left(1-e^{-a^2}\right)(2\pi)$$

$$= \pi\left(1-e^{-a^2}\right)$$

29/2/24

Q. $\displaystyle\int_{y=0}^{\frac{a}{\sqrt{2}}}\int_{x=y}^{\sqrt{a^2-y^2}} \log_e(x^2+y^2)\,dx\,dy$

Ans. $I = \displaystyle\int_{y}^{\frac{a}{\sqrt{2}}} \log_e(x^2+y^2)\,dx\,dy$

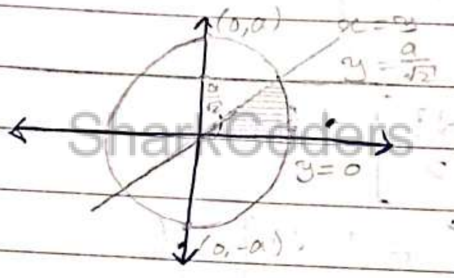
The curve is $y=0$ $\quad y=\frac{a}{\sqrt{2}}$,

$x^2+y^2=a^2$. $\quad x=y \quad x=\sqrt{a^2-y^2}$

$\Rightarrow x^2=a^2-y^2$

$x^2+y^2=a^2$.

$\therefore$ circle centered at (0,0) with radius $r=a$.



Consider strip ∥ to x-axis (integrate first with x)

limit(x) = $y=0$ $\quad y=\frac{a}{\sqrt{2}}$.

$x=0$ $\quad x=\frac{a}{\sqrt{2}}$

By using polar co-ordinate systems.

$x=r\cos\theta$

$y=r\sin\theta$

$x^2+y^2=r^2$

$dx\,dy = r\,dr\,d\theta$

$r: 0-a$

$\theta: 0-\pi/4$

29/2/24

$$I = \int_{\theta=0}^{\pi/4}\int_{r=0}^{a} \log r^2 \; r\,dr\,d\theta$$

$$= \int_0^{\pi/4} d\theta \left[\int_{r=0}^{a} 2\log r \; r\,dr\right]$$

$$= [\theta]_0^{\pi/4} \left(\log r \cdot \frac{r^2}{2} - \int \frac{1}{r}\cdot\frac{r^2}{2}\,dr\right)$$

$$= 2\left(\frac{\pi}{4} - 0\right)\left[\log r \cdot \frac{r^2}{2} - \frac{r^2}{4}\right]_0^a$$

$$= 2\left(\frac{\pi}{4} - 0\right)\left[\left(\frac{a^2}{2}\log a - \frac{a^2}{4}\right) - 0\right]$$

$$= \frac{2\pi}{4}\left[\frac{a^2}{2}\left(\log a - \frac{1}{2}\right)\right]$$

$$= \frac{\pi a^2}{4}\left(\log a - \frac{1}{2}\right)$$

Q. $\displaystyle\int_0^{a/\sqrt{2}}\int_x^{\sqrt{a^2-x^2}} \frac{x\,dx\,dy}{x^2+y^2}$

Ans. $\displaystyle I = \int_{x=0}^{a/\sqrt{2}}\int_{y=x}^{\sqrt{a^2-x^2}} \frac{x\,dx\,dy}{x^2+y^2}$

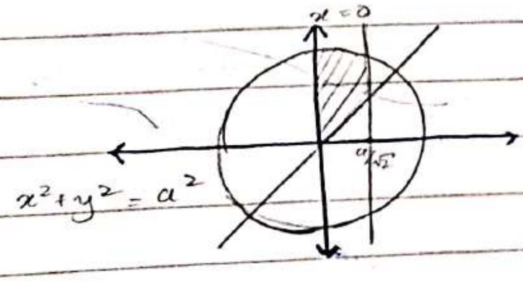
The curve is $x = 0$ $\quad$ $x = a/\sqrt{2}$

$y = x$ $\quad$ $y = \sqrt{a^2-x^2}$

$y^2 = a^2 - x^2$

$x^2 + y^2 = a^2$

∴ There is a circle of radius a centered at (0,0).



29/2/24

By using polar co-ordinates,

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2 = a^2$

$dx\,dy = r\,dr\,d\theta$

$r : 0 - a$

$\theta : \pi/2 - \pi/4$

$$I = \int_{\theta=\pi/2}^{\pi/4} \int_{r=0}^{a} \frac{r\cos\theta}{r^2}\, r\,dr\,d\theta.$$

$$= \int_{\pi/2}^{\pi/4} \left[\int_0^a \cos\theta\, dr \right] d\theta$$

$$= \left[\int_{\pi/2}^{\pi/4} \cos\theta\, d\theta \right] \left[\int_{r=0}^{a} dr \right]$$

$$= [r]_0^a \, [\sin\theta]_{\pi/4}^{\pi/2}$$

$$= (a-0)(-\sin\pi/4 + \sin\pi/2)$$

$$= a\left(-\frac{\sqrt{2}}{2} + 1\right)$$

$$= a\left(1 - \frac{1}{\sqrt{2}}\right).$$

27/2/24

Q. $\int_{x=0}^{2a}\int_{r=0}^{\sqrt{2ax-x^2}} (x^2+y^2)\, dx\, dy.$

Ans. $I = \int_{x=0}^{2a}\int_{r=0}^{\sqrt{2ax-x^2}} (x^2+y^2)\, dx\, dy.$

The curve is $x = 0$ $\quad x = 2a$

$y = 0$ $\quad y = \sqrt{2ax-x^2}$

$y^2 = 2ax - x^2$

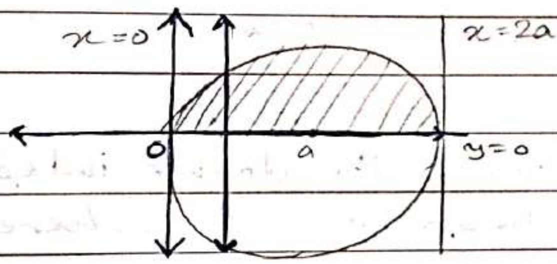
$x^2+y^2 = 2ax$

$x^2 - 2ax + y^2 = 0$

$x^2 - 2ax + a^2 + y^2 - a^2 = 0$

$(x-a)^2 + y^2 = a^2$

The circle with radius a is centered at $(a, 0)$.



By using polar co-ordinates,

$x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow x^2+y^2 = r^2 = 2ax = 2ar\cos\theta \quad r = 2a\cos\theta$

$dx\, dy = r\, dr\, d\theta.$

$I =$ $\quad r: 0 - 2a\cos\theta$

$\theta: 0 - \pi$

29/2/24

$$I = \int_{\theta=0}^{\pi}\int_{r=0}^{2a\cos\theta} 2ar\cos\theta \, dr \, d\theta$$

$$= \left[\int_{\theta=0}^{\pi} d\theta\right]\left[\int_{r=0}^{2a\cos\theta} 2ar\cos\theta \cdot r \, dr\right]$$

$$= \frac{1}{4}\int_{\theta=0}^{a} \left[(2a\cos\theta)^4 - 0\right] d\theta$$

$$= \frac{1}{4} \, 16a^4 \int_{0}^{\pi} \cos^4\theta \, d\theta$$

$$I = 4a^4 \int_{\theta=0}^{\pi} \cos^4\theta \, d\theta = 4a^4 \cdot 2\int_{0}^{\pi/2} \cos^4\theta \, d\theta$$

$$= 8a^4 \left[\frac{3}{4} \cdot \frac{1}{2}\right]\frac{\pi}{2}$$

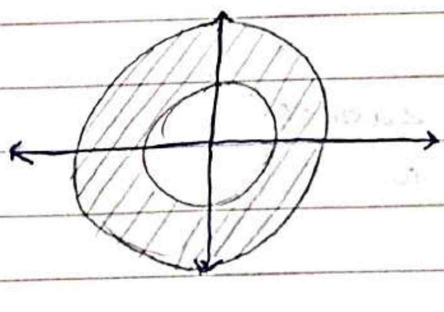
$$= \frac{3a^4\pi}{2}$$

Type IV: Evaluating the double integration using polar co-ordinate system where limit is not given

Q. $\iint_R \frac{x^2y^2}{x^2+y^2} \, dx \, dy$ where R is annulus

$R: x^2+y^2=4$ and $x^2+y^2=9$

Ans. $I = \iint_R \frac{x^2y^2}{(x^2+y^2)} \, dx \, dy.$



29/2/24

Limit(r) : 2 - 3

θ : 0 - 2π

$$I = \int_{\theta=0}^{2\pi} \int_{r=2}^{3} \frac{(r\cos\theta)^2 (r\sin\theta)^2}{r^2} \cdot r \, dr \, d\theta$$

$$= \left[\int_{\theta=0}^{2\pi} \cos^2\theta \cdot \sin^2\theta \, d\theta\right]\left[\int_{r=2}^{3} r^3 \, dr\right]$$

$$= \left[4\int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta\right]\left[\frac{r^4}{4}\right]_2^3$$

$$= \left\{\cancel{4}\left[\frac{1 \cdot 1}{\cancel{4} \cdot 2}\right]\frac{\pi}{2}\right\}\left[\frac{81}{4} - \frac{16}{4}\right]$$

$$= \frac{\pi}{4}\left(\frac{65}{4}\right)$$

$$= \frac{65\pi}{16}$$

1/3/24 ~~[illegible]~~

Q. Type IV: $\iint \sin(x^2+y^2) \, dx \, dy$

R : $x^2 + y^2 = a^2$ 3

r : 0 to a

θ : 0 to 2π

Using polar co-ordinates,

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2 = a^2$

$dx\,dy = r\,dr\,d\theta$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} \sin r^2 \cdot r \, dr \, d\theta$$

1/3/24

$$I = \left[4\int_{\theta=0}^{\pi/2} d\theta\right]\left[\int_{r=0}^{a} \sin r^2 \cdot r\, dr\right]$$ $r^2 = t$

$r^2 = t$

$2r\,dr = dt$

$r\,dr = \dfrac{dt}{2}$

r	0	to	a
t	0	to	a^2

$$I = 4\,[\theta]_0^{\pi/2} \cdot \left[\int_{r=0}^{a^2} \frac{\sin t}{2}\, dt\right]$$

$$= 4(\pi/2)\left(\frac{1}{2}\right)\left[\cos t\right]_0^{a^2}$$

$$= \pi(\cos a^2 - 1)$$

Q. Evaluate $\iint_R y^2\,dx\,dy$ over the area which lies outside the curve $x^2+y^2=ax$ and lies inside the curve $x^2+y^2=2ax$

Ans. $x^2+y^2=2ax$

$x^2-2ax+a^2+y^2=0$

$(x-a)^2+y^2=a^2$

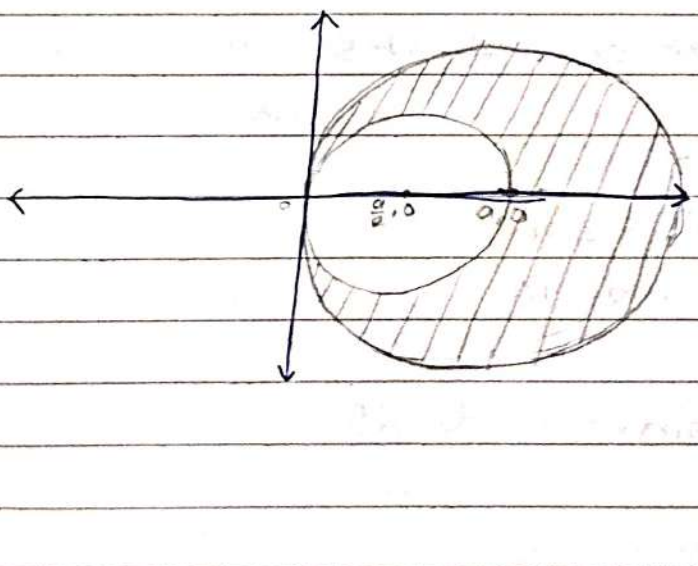
$C\in(a,0)$

$x^2+y^2=ax$

$x^2+ax+\frac{a}{4}+y^2=\frac{a}{4}$

$(x-\frac{a}{2})^2+y^2=\frac{a}{4}$

$C\in(\frac{a}{2},0)$



1/3/24

$x^2+y^2=r^2 = ar\cos\theta \Rightarrow r = a\cos\theta$

$x^2+y^2=r^2 = 2ar\cos\theta \Rightarrow r = 2a\cos\theta$

$r : a\cos\theta$ to $2a\cos\theta$

$\theta : -\pi/2$ to $\pi/2$

$y^2 = 2ar\cos\theta - r^2\cos^2\theta$

$= r\cos\theta(2ar - r\cos\theta)$

$x^2+y^2 = r^2$

$r\,dr\,d\theta = dx\,dy$

$$I = \int_{\theta=-\pi/2}^{\pi/2}\int_{r=a\cos\theta}^{2a\cos\theta} y^2\, r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2}\int_{r=a\cos\theta}^{2a\cos\theta} r^3\sin^2\theta\, dr\, d\theta$$

$$= \int_{r=a\cos\theta}^{2a\cos\theta}\left[\frac{r^4}{4}\right] dr \cdot 2\int^{\pi/2}\sin^2\theta\, d\theta$$

$$= \left[\frac{r^4}{4}\right]_{a\cos\theta}^{2a\cos\theta} \cdot 2\int_{\theta=0}^{\pi/2}\sin^2\theta\, d\theta$$

$$= 2\int_0^{\pi/2}\frac{(16a^4\cos^4\theta - a^4\cos^4\theta)}{4}\sin^2\theta\, d\theta$$

$$= \frac{15}{2}\int_{\theta=0}^{\pi/2}\frac{a^4\cos^4\theta\sin^2\theta}{4}\, d\theta$$

$$= \frac{15a^4}{2}\int_{\theta=0}^{\pi/2}\sin^2\theta\cos^4\theta\, d\theta$$

$$= \frac{15a^4}{2}\left[\frac{(1)\cdot(3)(1)}{(6)(4)(2)(1)}\right]\frac{\pi}{2}$$

$$= \frac{15a^4}{2}\left(\frac{1}{16}\right)$$

$$= \frac{15a^4\pi}{64}$$

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Q. Evaluate $\iint_R \frac{\sqrt{x^2+y^2}}{x^2}\,dx\,dy$ where R is a region bounded by $x^2+y^2=2x$, $y=x$, $y=0$. and in first quadrant.

Ans. $I = \iint_R \frac{\sqrt{x^2+y^2}}{x^2}\,dx\,dy$

$R: x^2+y^2=2x$

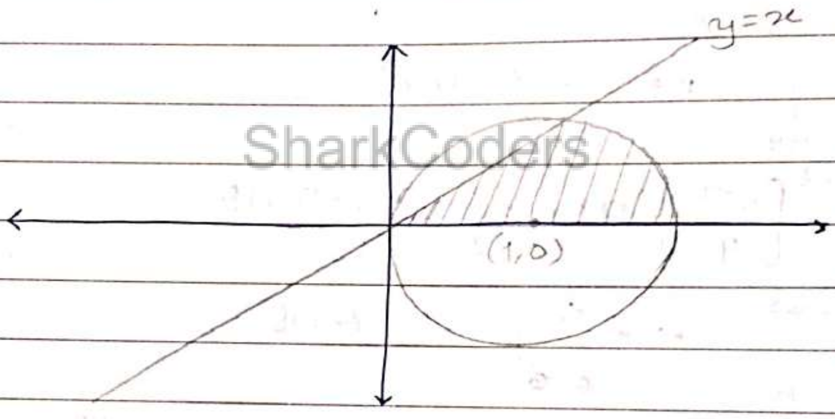
$x^2-2x+1^2+y^2=1^2$

$(x-1)^2+y^2=1^2$

$C \in (1,0)$

$y=x$

$y=0$



By using polar co-ordinates,

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2 = 2r\cos\theta$

$r = 2\cos\theta$

$dx\,dy = r\,dr\,d\theta$

$r: 0$ to $2\cos\theta$

$\theta: 0$ to $\pi/4$

$$I = \int_{\theta=0}^{\pi/4} \int_{r=0}^{2\cos\theta} \frac{r^2}{r^2\cos^2\theta} \, r \, dr \, d\theta = \int_{\theta=0}^{\pi/4} \int_{r=0}^{2\cos\theta} \frac{1}{\cos^2\theta} \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/4} \left[\frac{r}{\cos^2\theta} \right]_0^{2\cos\theta} d\theta$$

$$= \int_{\theta=0}^{\pi/4} \left[\frac{r}{\cos^2\theta} \right]_0^{2\cos\theta} d\theta$$

$$= \int_{\theta=0}^{\pi/4} \left[\frac{2\cos\theta}{\cos^2\theta} \right] d\theta$$

$$= \int_{\theta=0}^{\pi/4} \frac{2}{\cos\theta} \, d\theta$$

$$= 2 \int_{\theta=0}^{\pi/4} \sec\theta \, d\theta$$

$$= 2 \left[\log(\sec\theta + \tan\theta) \right]_0^{\pi/4}$$

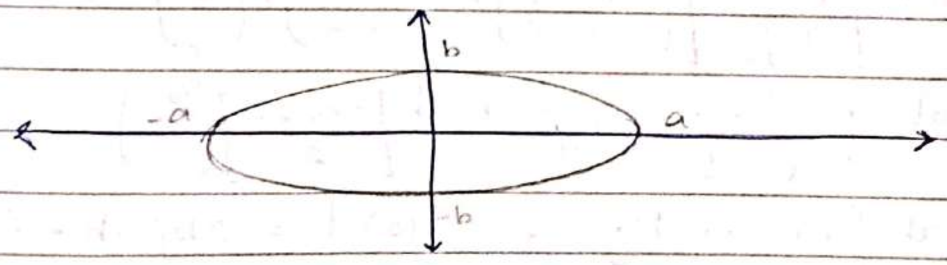
$$= 2 \left[\log(\sec(\pi/4) + \tan(\pi/4)) - \log(\sec(0) + \tan(0)) \right]$$

$$= 2 \left[\log(\sqrt{2} + 1) - \log 1 \right]$$

$$= \cancel{2 \log \log (2} \quad 2 \log(\sqrt{2} + 1)$$

Q. Evaluate $\iint_R (x+y)^2 \, dxy$ where area is bounded by area $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ans.



By using polar co-ordinates,

$x = a r\cos\theta$

$y = b r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.

$r : 0$ to 1
$\theta : 0$ to 2π

$$(x+y)^2 = x^2+y^2+2xy$$
$$= r^2(a^2+b^2) + 2r^2 ab \cos\theta \sin\theta$$

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^2(a^2+b^2) + 2r^2 ab \sin\theta \cos\theta$$
$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^2a^2 + r^2b^2 + 2r^2 ab \sin\theta \cos\theta \; dr\, d\theta$$
$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^2(a^2+b^2+2ab\sin\theta\cos\theta)\, abr\, dr\, d\theta$$
$$= ab \int_{\theta=0}^{2\pi} \cdot \int_{r=0}^{1} r^3 (a^2+b^2+2ab\sin\theta\cos\theta)\, dr\, d\theta$$

$$(x+y)^2 = r^2[a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta]$$

$$I = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^2[a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta]\, abr\, dr\, d\theta$$
$$= 4ab\left[\int_{\theta=0}^{\pi/2}(a^2\cos^2\theta + b^2\sin^2\theta + 2ab\sin\theta\cos\theta)\, d\theta\right]\left[\int_{r=0}^{1} r^3\, dr\right]$$
$$= 4ab\left[\frac{r^4}{4}\right]_0^1 \left[\left(a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) + \left(b^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) + \left(ab\int_0^{\pi/2}\sin 2\theta\, d\theta\right)\right]$$
$$= 4ab\left(\frac{1}{4}\right)\left(\frac{a^2\pi}{4} + \frac{b^2\pi}{4} + ab\left[\frac{\cos 2\theta}{2}\right]_0^{\pi/2}\right)$$
$$= ab\left[\frac{a^2\pi}{4} + \frac{b^2\pi}{4} + \frac{ab(2)}{2}\right] = ab\left(ab + \frac{(a^2+b^2)\pi}{4}\right)$$

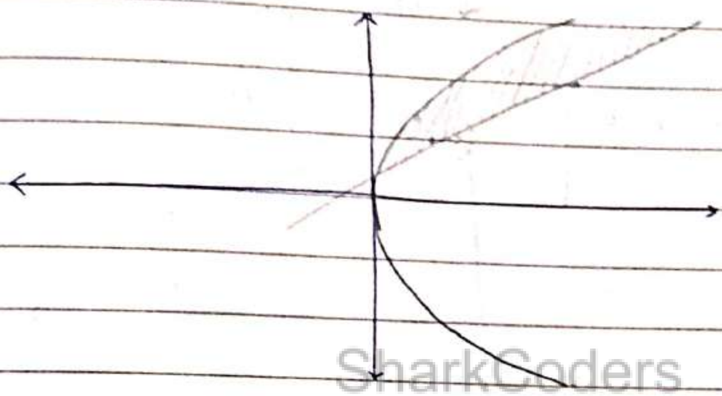
Type V : Area in cartesia co-ordinate,

$$A = \iint dx\, dy$$

In polar co-ordinates

$$A = \iint r\, dr\, d\theta$$

Find the area bounded by the parabola $y^2 = 4x$ and the line $2x - 3y + 4 = 0$.



$y^2 = 4x$	$2(2x - 3y + 4 = 0)$
$16 = 4x$	$4x - 6y + 8 = 0$
$x = 4$	$y^2 - 6y + 8 = 0$
$4 = 4x$	$y = 4, 2$
$x = 1$.	

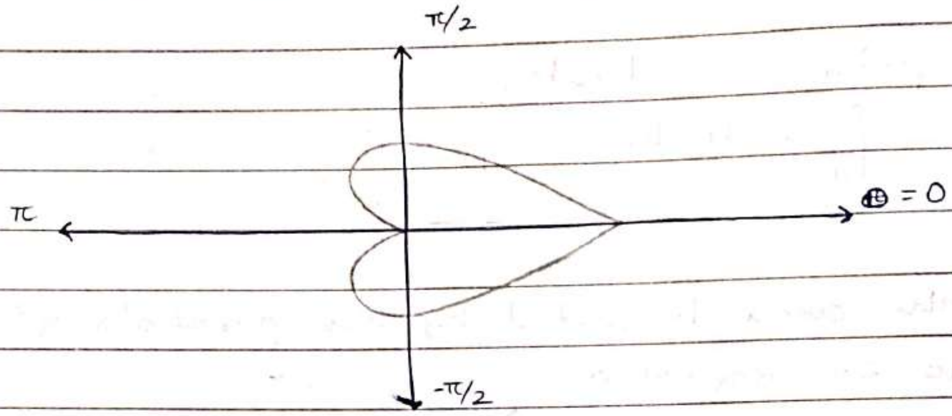
$\therefore$ $x = 1$, $y = 2$ $\qquad$ $x = 4$, $y = 4$

Consider

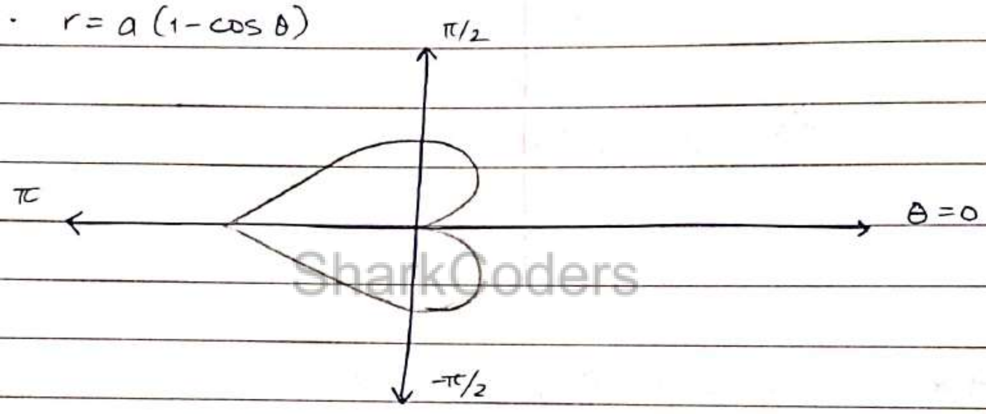
5/3/24

Cardioids

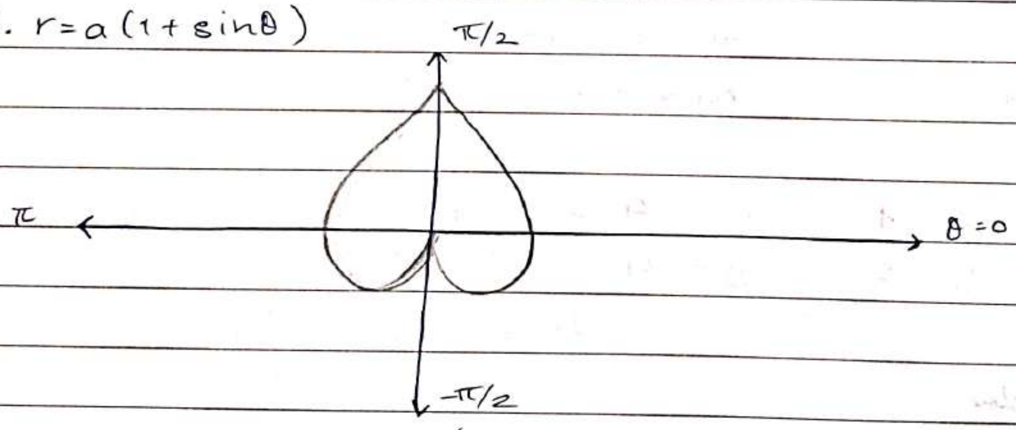
1. $r = a(1+\cos\theta)$



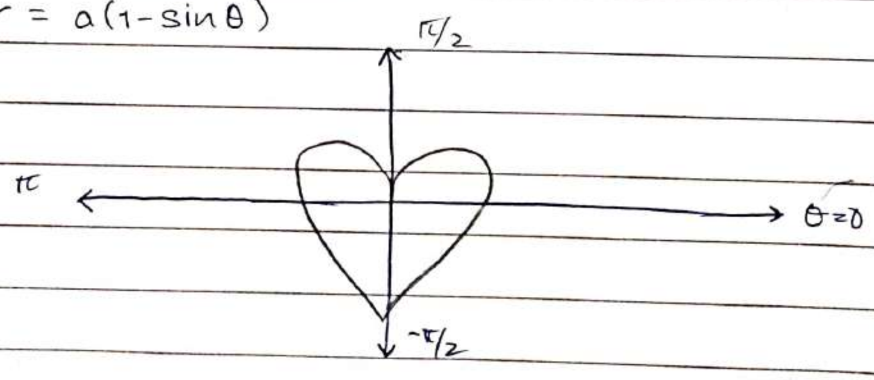
2. $r = a(1-\cos\theta)$



3. $r = a(1+\sin\theta)$



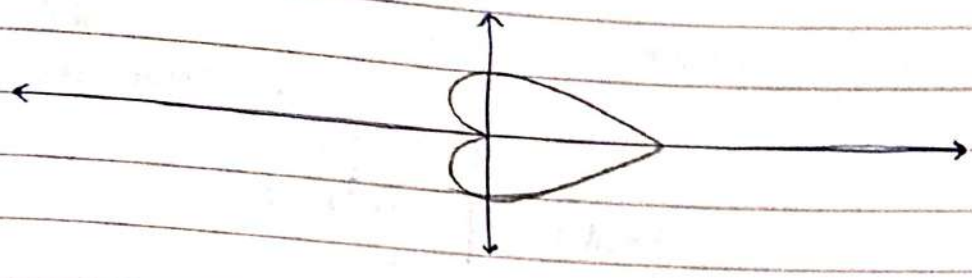
4. $r = a(1-\sin\theta)$



Q. Find the area of cardioid $r = a(1+\cos\theta)$

Ans $r = a(1+\cos\theta)$

In polar co-ordinates



$$A = \iint r\,dr\,d\theta$$

r : 0 to $a(1+\cos\theta)$

θ : 0 to 2π

$$A = \int_{r=0}^{a(1+\cos\theta)} \int_{\theta=0}^{2\pi} r\,dr\,d\theta$$

$$= \cancel{\int_{\theta=0}^{2\pi} \left[\frac{r^2}{2}\right]_0^{2\pi} d\theta} = \int_{\theta=0}^{2\pi} \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \cancel{\int_{\theta=0}^{2\pi} \left(\frac{4\pi^2}{2}\right) d\theta} \quad \cancel{= \int_{\theta=0}^{2\pi} 2\pi^2\, d\theta}$$

$$\cancel{= 2\pi^2(2\pi)}$$

$$= \int_{\theta=0}^{2\pi} \frac{a^2(1+\cos^2\theta+2\cos\theta)}{2}\, d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{2\pi} (1+\cos^2\theta+2\cos\theta)\, d\theta$$

$$= \frac{a^2}{2} \left\{ \left[\theta + 2\sin\theta\right]_0^{2\pi} + 4\int_0^{\pi/2} \cos^2\theta\, d\theta \right\}$$

$$= \frac{a^2}{2} \left[2\pi + 2\sin(2\pi) + 4\left[\frac{(1)}{(2)} \cdot \frac{\pi}{2}\right] \right]$$

$$= \frac{a^2}{2}(3\pi) = \frac{3a^2\pi}{2}$$

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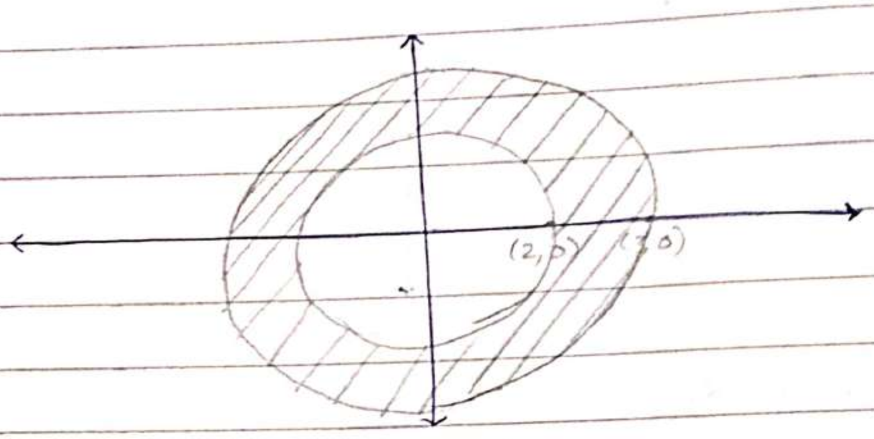
Q. Find the area of annulus result between the circle $x^2+y^2=4$ and $x^2+y^2=9$.



By polar co-ordinate system,

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$

$dx\,dy = r\,dr\,d\theta$

$r : 2$ to 3

$\theta : 0$ to 2π

$$\int_{\theta=0}^{2\pi}\int_{r=2}^{3} r\,dr\,d\theta = \int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_2^3 d\theta$$

$$= \int_{\theta=0}^{2\pi}\left(\frac{9}{2}-\frac{4}{2}\right)d\theta$$

$$= \frac{5}{2}\left[\theta\right]_0^{2\pi}$$

$$= \frac{10\pi}{2} = 5\pi$$

Q. Find the area which lies outside the circle $x^2+y^2=ax$ and inside the circle $x^2+y^2=2ax$.

Ans

$x^2+y^2=ax$

$x^2-ax+\frac{a^2}{4}+y^2=\frac{a^2}{4}$

$(x-\frac{a}{2})^2+y^2=\frac{a^2}{4}$

$C \in (a/2, 0)$

$r = a/4$

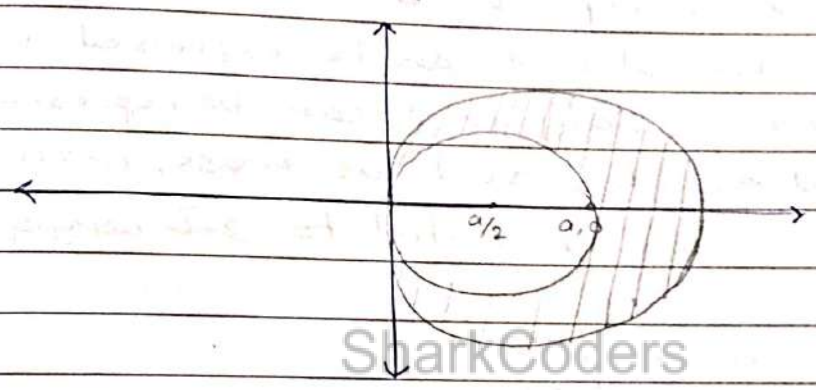
$x^2+y^2=2ax$

$x^2-2ax+a^2+y^2=a^2$

$(x-a)^2+y^2=a^2$

$C \in (a, 0)$

$r = a$



Sol:

$x^2+y^2=r^2=ax$

$r^2=ar\cos\theta$

$r=a\cos\theta$

$x^2+y^2=2ax=r^2$

$r^2=2ar\cos\theta$

$r=2a\cos\theta$

$r: a\cos\theta$ to $2a\cos\theta$

$\theta: -\pi/2$ to $\pi/2$

$$I=\int_{\theta=-\pi/2}^{\pi/2}\int_{r=a\cos\theta}^{2a\cos\theta} r\,dr\,d\theta = \int_{\theta=-\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_{a\cos\theta}^{2a\cos\theta} d\theta$$

$$=\int_{\theta=-\pi/2}^{\pi/2}\left(\frac{4a^2\cos^2\theta - a^2\cos^2\theta}{2}\right)d\theta$$

$$=\int_{\theta=-\pi/2}^{\pi/2}\frac{3a^2\cos^2\theta}{2}\,d\theta = \frac{3a^2\cdot 2}{2}\int_{\theta=0}^{\pi/2}\cos^2\theta\,d\theta$$

$$=3a^2\left[\frac{(1)}{(2)}\cdot\frac{\pi}{2}\right]=\frac{3\pi a^2}{4}$$

1. Cartesian – with limit
   – without limit.
2. Polar – with limit
   – without limit
3. Area – cartesian co-ordinate
   – polar coordinate

Triple Integration

Extended concept of double integration to 3rd dimension. Basically, it can be expressed in terms of volume of body or it can be represented in any physical quantity such as mass, moment of inertia, center of gravity related to 3-D configuration.

NOTE :- use of notation

1) $$\int_{x=a}^{b}\int_{y=f_2(x)}^{f_1(x)}\int_{z=\phi_1(x,y)}^{\phi_2(x,y)} (x,y,z)\, dz\, dy\, dx$$

$$\int_{x=a}^{b}\left(\int_{y=f_1(x,y)}^{f_2(x,y)}\left\{\int_{z=\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\, dz\right\} dy\right) dx.$$

- the order of integration depends upon distribution of limit.
- use of spherical polar co-ordinate spherical polar coordinate are written by

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$x^2+y^2+z^2=r$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2\sin\theta$$

5/3/24

$$dx\,dy\,dz = r^2 \sin\theta \cdot dr\, d\theta\, d\varphi$$

$$I = \iiint_V f(r,\theta,\varphi)\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

Standard limit

1. For complete sphere,

$x^2+y^2+z^2=a^2$

θ : 0 to 2π

φ : 0 to 2π

r : 0 to a

2. Hemisphere

$x^2+y^2+z^2=a^2$ where $z>0$

θ : 0 to $\pi/2$

φ : 0 to 2π

r : 0 to a

3. Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

θ : 0 to π

φ : 0 to 2π

r : 0 to 1

Dirichlet's Theorem

If $x+y+z \le 1$, then

$$\iiint x^{a-1} y^{b-1} z^{c-1}\, dx\, dy\, dz = \frac{\Gamma a\, \Gamma b\, \Gamma c}{\Gamma 1+a+b+c}$$

6/3/24

Q. $\int_{y=0}^{1} dy \int_{x=y^2}^{1} dx \int_{z=0}^{1-x} x \, dz =$

Ans. $I = \int_{y=0}^{1} dy \int_{z=y^2}^{1} dx \int_{z=0}^{1-x} x \, dz =$

$= \int_{y=0}^{x=y^2} dy \int_{x=y^2}^{1} dx \; x[z]_0^{1-x} =$

$\int_{y=0}^{1} dy \int_{x=y^2}^{1} (x - x^2) \, dx =$

$\int_{y=0}^{1} \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{y^2}^{1} dy =$

$\int_{y=0}^{1} \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{y^4}{2} - \frac{y^6}{3}\right) dy =$

$\int_{y=0}^{1} \frac{1 + 2y^6 - 3y^4}{6} \, dy = \frac{1}{6} \int_{y=0}^{1} 1 + 2y^6 - 3y^4 \, dy$

$35 + 10 - 21$

$= \frac{1}{6}\left[y + \frac{2y^7}{7} - \frac{3y^5}{5}\right]_0^1 = \frac{1}{6}\left[1 + \frac{2}{7} - \frac{3}{5}\right] = \frac{1}{6} \cdot \frac{24}{35}$

$= \frac{4}{35}$

Q. $\int_{x=0}^{2} dx \int_{y=0}^{x} dy \int_{z=y}^{x} xyz \, dz.$

Ans. $I = \int_{x=0}^{2} dx \cdot \int_{y=0}^{x} dy \cdot \int_{z=y}^{x} xyz \, dz =$

$\int_{x=0}^{2} dx \cdot \int_{y=0}^{x} dy \cdot xy[z]_y^x =$

$\int_{x=0}^{2} dx \cdot \int_{y=0}^{x} x^2y - xy^2 \, dy =$

6/3/24

$$\int_{x=0}^{2} dx \cdot \left[\frac{x^2y^2}{2} - \frac{xy^3}{3}\right]_0^x =$$

$$\int_{x=0}^{2} \left(\frac{x^4}{2} - \frac{x^4}{3}\right) dx = \int_{x=0}^{2} \frac{x^4}{6}\, dx$$

$$= \cancel{\left[\frac{x^4}{6}\right]_0^2} \quad \frac{1}{6}\left[\frac{x^5}{5}\right]_0^2$$

$$= \cancel{\frac{16}{6} = \frac{8}{3}} \quad \frac{1}{6}\left(\frac{32}{5}\right) = \frac{16}{15} = \frac{4}{5}$$

Q. $$\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{x+y} e^z\, dz\, dy\, dx$$

Ans. $$I = \int_{x=0}^{1}\int_{y=0}^{1-x} [e^z]_0^{x+y}\, dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} e^{x+y} \cdot e^{y-1}\, dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} e^x \cdot e^y - y\, dy\, dx$$

$$= \int_{x=0}^{1} \left[e^x \cdot e^y - y\right]_0^{1-x} dx = \int_{x=0}^{1} \left[e^x \cdot e^y - y\right]_0^{1-x} dx$$

$$= \cancel{\int_{x=0}^{1} (e^x \cdot e^{(1-x)} + x - 1)\, dx} = \int_{x=0}^{1} \left(e^x \cdot e^{1-x} - (1-x)\right) - \left(e^x \cdot e^0 - 0\right) dx$$

$$= \cancel{\int_{x=0}^{1} \left(\frac{e^x \cdot e}{e^x} + x - 1\right) dx} = \int_{x=0}^{1} \left(\frac{e^x e}{e^x} + x - 1 - e^x\right) dx$$

$$= \cancel{\left[ex + \frac{x^2}{2} - x\right]_0^1} = \left[ex + \frac{x^2}{2} - x - e^x\right]_0^1$$

$$= \cancel{\left(e + \frac{1}{2} - 1\right) - (0)} = \left(e + \frac{1}{2} - 1 - e\right) - \left(-e^0\right)$$

$$= e + \frac{1}{2} - 1 - e + 1$$

$$= \frac{1}{2}$$

6/3/24

Q. Evaluate $\iiint z(x^2+y^2)\,dx\,dy\,dz$ over the volume of the cylinder $x^2+y^2=1$ intercepted by the plane $z=2$ and $z=3$.

Ans. $I = \iint \int_{z=2}^{3} z(x^2+y^2)\,dz\,dy\,dx$

$= \iint (x^2+y^2)\left[\frac{z^2}{2}\right]_2^3 dy\,dx$

$= \iint (x^2+y^2)\left(\cancel{[illegible]}\,\frac{5}{2}\right) dy\,dx$

$= \frac{5}{2}\iint x^2+y^2\,dy\,dx$

Using polar co-ordinates,

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2 = 1 \Rightarrow r=1$

$dx\,dy = r\,dr\,d\theta$

$r : 0$ to 1

$\theta : 0$ to 2π

$\frac{5}{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\,r\,dr\,d\theta = \left[\int_{\theta=0}^{2\pi} d\theta\right]\cdot\left[\frac{r^4}{4}\right]_0^1$

$= \frac{(2\pi)}{4}\left(\frac{1}{8}\right) = \frac{5\pi}{\cancel{8}4}$

7/3/24

~~Q. $z = x^2+y^2$ to $\sqrt{x^2+y^2}$~~

~~$z = 1$~~

~~$x^2+y^2=1$~~

~~$V = \iiint$~~

Volume

1. Let V be the volume of solid can be expressed as a triple integration

$$V = \iiint dx\, dy\, dz$$

2. In spherical polar co-ordinate

$$V = \iiint r^2 \sin\theta \; dr\, d\theta\, d\phi$$

3. In cylindrical co-ordinate

$$V = \iiint r\, dr\, d\phi\, dz$$

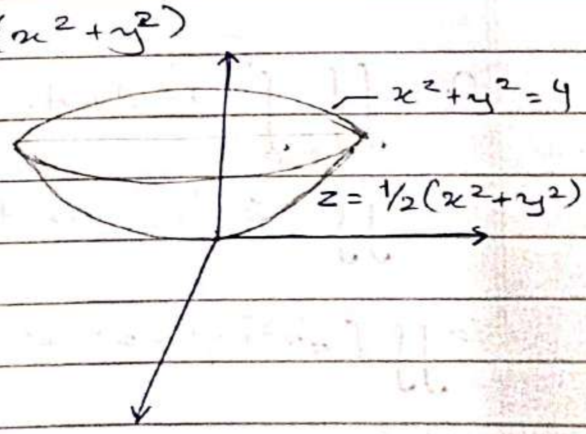
Example

Find the volume of the region bounded by the paraboloid $x^2+y^2=2z$ and the cylinder $x^2+y^2=4$

Volume $V = \iiint dx\, dy\, dz$

$V: x^2+y^2 = 2z$ $\qquad x^2+y^2=4$

limit of z: $z=0$ to $z = \frac{1}{2}(x^2+y^2)$



$$V = \iint \int_{z=0}^{\frac{1}{2}(x^2+y^2)} dz\, dx\, dy$$

$$= \iint [z]_0^{\frac{1}{2}(x^2+y^2)} dx\, dy$$

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$$V = \frac{1}{2}\iint (x^2+y^2)\,dx\,dy$$

$x^2+y^2=4$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$

$dx\,dy = r\,dr\,d\theta$

$r : 0$ to 2

$\theta : 0$ to 2π

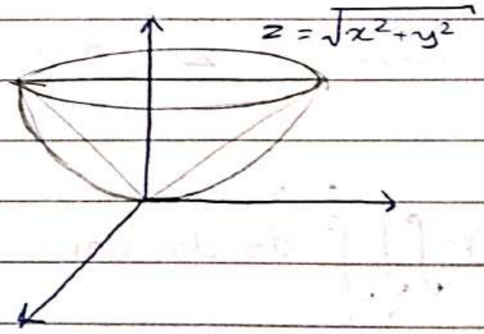
$$V = \frac{1}{2}\int_0^{2\pi}\int_0^2 r^2\,dr \cdot r\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta$$

$$= \frac{16}{8}\int_0^{2\pi} d\theta = 2(2\pi) = 4\pi$$

Q. Find the volume of the region inclosed by the cone $z=\sqrt{x^2+y^2}$ and paraboloid $z=x^2+y^2$

Ans. $z = x^2+y^2$ to $\sqrt{x^2+y^2}$



$$V = \iint\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz\,dx\,dy$$

$$= \iint [z]_{x^2+y^2}^{\sqrt{x^2+y^2}}\,dx\,dy$$

$$= \iint \left[\sqrt{x^2+y^2} - x^2 - y^2\right] dx\,dy$$

Using polar co-ordinates,

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$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

$\sqrt{x^2+y^2} = r$

$x^2 + y^2 = (\sqrt{x^2+y^2})^2$

$=$.

$\sqrt{x^2+y^2} = 1$

$r = 1$

~~limits~~ $r : 0$ to 1

$\theta : 0$ to 2π

$$V = \int_0^{2\pi}\int_0^1 [r - r^2]\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 [r^2 - r^3]\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3} - \frac{r^4}{4}\right]_0^1 d\theta$$

$$= [\theta]_0^{2\pi} \left[\frac{1}{3} - \frac{1}{4}\right]$$

$$= 2\pi\left[\frac{1}{12}\right] = \frac{\pi}{6}$$

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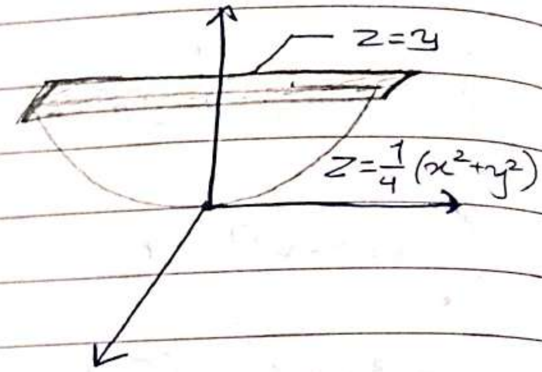
Q. Find the volume of paraboloid of revolution $x^2+y^2=4z$ cut off by the plane $z=y$



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Q. Find the volume of tetrahedron bounded by the co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

~~Ans.~~

Ans $V: \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Putting $x = a\,dy,\ y = b\,dv,\ z = c\,dw$

$V = \iiint abc\ du\ dv\ dw$

$u + v + w = 1$

By Dirichelt's theore

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Q. Find the vol. of tetrahedron and bounded by. cordinate :

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Vector Calculus

Notes

- $a \cdot b = |a||b|\cos\theta$
  → dot product
  ~~$a \times b = |a||b|\sin\theta$~~
  → $\vec{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$
  $\vec{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

- $a \times b = |a||b|\sin\theta$

- $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\frac{d\bar{r}}{dt}$ ⟶ tangent vector (geometrical interpretation)
⟶ velocity (physical interpretation)

Type 1: Based on velocity and ~~acc~~ acceleration

Q. A particle is moving along the curve $\bar{r} = e^{-t}\bar{i} + e^{2t}\bar{j} + e^{t}\bar{k}$
Find the velocity and acceleration at $t = 0$

Ans. Given

$\bar{r} = e^{-t}\bar{i} + e^{2t}\bar{j} + e^{t}\bar{k}$ (~~position vector~~)

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ (position vector)

$$\text{velocity} = \bar{v} = \frac{d\bar{r}}{dt} =$$

$$= \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$$

$$= \frac{d(e^{-t})}{dt}\bar{i} + \frac{d(e^{2t})}{dt}\bar{j} + \frac{d(e^{t})}{dt}\bar{k}$$

$$= -e^{-t}\bar{i} + 2e^{2t}\bar{j} + e^{t}\bar{k}$$

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$[v]_{t=0} = -\bar{i} + 2\bar{j} + \bar{k}$

acceleration $= \bar{a} = \dfrac{d\bar{v}}{dt}$

$\bar{a} = \dfrac{d(-e^{-t})}{dt}\bar{i} + \dfrac{2d(e^{2t})}{dt}\bar{j} + \dfrac{d(e^{t})}{dt}\bar{k}$

$= e^{-t}\bar{i} + 4e^{2t}\bar{j} + e^{t}\bar{k}$

$[\bar{a}]_{t=0} = \bar{i} + 4\bar{j} + \bar{k}$

Q. Find the magnitude of velocity and acceleration of a particle which moves along the curve $x = 2\sin(3t)$, $y = 2\cos(3t)$, $z = 8t$ at any time $t > 0$.

Ans. Given

$x = 2\sin(3t)$

$y = 2\cos(3t)$

$z = 8t$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ (position vector)

$= 2\sin(3t)\bar{i} + 2\cos(3t)\bar{j} + 8t\bar{k}$

$\bar{v} = \dfrac{d(2\sin 3t)}{dt}\bar{i} + \dfrac{d(2\cos 3t)}{dt}\bar{j} + \dfrac{d(8t)}{dt}\bar{k}$

$= [2(\cos(3t)\cdot 3]\bar{i} - 2\sin(3t)\cdot 3\bar{j} + 8\bar{k}$

$= 6\cos 3t\,\bar{i} - 6\sin 3t\,\bar{j} + 8\bar{k}$

magnitude

$|\bar{v}| = \sqrt{(6\cos(3t))^2 + (-6\sin(3t))^2 + (8)^2}$

$= \sqrt{36\cos^2 3t + 36\sin^2 3t + 64}$

$= \sqrt{36 + 64} = 10$

$\therefore |v| = 10$

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$$\bar{a} = \frac{d\bar{v}}{dt}$$

$$= \frac{d(6\cos 3t)}{dt}\bar{i} + \frac{d(\cancel{6}\sin 3t)}{dt}\bar{j} + \frac{d(8)}{dt}\bar{k}$$

$$= -6\sin 3t \cdot 3\bar{i} + 6\cos 3t \cdot 3\bar{j}$$

$$= (-18\sin 3t)\bar{i} + (-18\cos 3t)\bar{j}$$

$$|\bar{a}| = \sqrt{(-18\sin 3t)^2 + (-18\cos 3t)^2}$$

$$= \sqrt{324(1)}$$

$$\therefore |\bar{a}| = 18$$

H.W.

Q. For the curve $\bar{r} = e^{-t}\bar{i} + \log(t^2+1)\bar{j} - (\tan t)\bar{k}$. Find the velocity and acceleration at $t=0$.

Q. For the curve $x = e^t\cos t$, $y = e^t\sin t$, $z = e^t$. Find the velocity and acceleration of the particle moving on the curve at $t=0$.

Type II: angle between tangent to curve

Q. Find the angle between tangent to the curve, $x = t$, $y = t^2$, $z = t^3$ at $t=1$ and $t=-1$.

Ans. Given

$x = t$

$y = t^2$

$z = t^3$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ (position vector)

$\bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$

Tangent vector $= \bar{T} = \dfrac{d\bar{r}}{dt}$

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$$\bar{T} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$$

$$= \frac{d(t)}{dt}\bar{i} + \frac{d(t^2)}{dt}\bar{j} + \frac{d(t^3)}{dt}\bar{k}$$

$$\bar{T} = \bar{i} + 2t\bar{j} + 3t^2\bar{k}$$

$$[\bar{T}]_{t=1} = \bar{T}_1 = \bar{i} + 2\bar{j} + 3\bar{k}$$
$$[\bar{T}]_{t=-1} = \bar{T}_2 = \bar{i} - 2\bar{j} + 3\bar{k}$$

angle between $\bar{T}_1$ and $\bar{T}_2$ is

$$\cos\theta = \frac{\bar{T}_1 \cdot \bar{T}_2}{|\bar{T}_1| \cdot |\bar{T}_2|} \quad \text{—— (1)}$$

$$\bar{T}_1 \cdot \bar{T}_2 = (\bar{i} + 2\bar{j} + 3\bar{k}) \cdot (\bar{i} - 2\bar{j} + 3\bar{k})$$
$$= (1 \cdot 1) + (2 \cdot -2) + (3 \cdot 3)$$
$$= 1 - 4 + 9 = 6 \quad \text{—— (2)}$$

$$|\bar{T}_1| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \quad \text{—— (3)}$$
$$|\bar{T}_2| = \sqrt{1^2 + 4 + 3^2} = \sqrt{14} \quad \text{—— (4)}$$

Substituting (2), (3), and (4) in (1),

$$\bar{T}_1 \cdot \bar{T}_2 = \frac{6}{\sqrt{14} \cdot \sqrt{14}} = \frac{6}{14} = \frac{3}{7} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{3}{7}\right) = 64.62^\circ$$

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Q. Find the angle between tangent to the curve $t^2\bar{i} + 2t\bar{j} - t^3\bar{k} = \bar{r}$ at point $t = -1$ and $t = 1$.

Ans. Given

$\bar{r} = t^2\bar{i} + 2t\bar{j} - t^3\bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\bar{T} = \frac{d(t^2)}{dt}\bar{i} + \frac{d(2t)}{dt}\bar{j} + \frac{d(-t^3)}{dt}\bar{k}$$

$$= 2t\bar{i} + 2\bar{j} - 3t^2\bar{k}$$

$[\bar{T_1}]_{t=1} = 2\bar{i} + 2\bar{j} - 3\bar{k}$

$[\bar{T_2}]_{t=-1} = -2\bar{i} + 2\bar{j} - 3\bar{k}$

~~Tango~~ $\cos\theta = \dfrac{\bar{T_1}\cdot\bar{T_2}}{|\bar{T_1}|\cdot|\bar{T_2}|}$

$\bar{T_1}\cdot\bar{T_2} = (2)(-2) + (2)(2) + (-3)(-3)$

$= -4 + 4 + 9$

$= 9$

$|T_1| = \sqrt{4+4+9} = \sqrt{17}$

$|T_2| = \sqrt{4+4+9} = \sqrt{17}$

$\theta = \cos^{-1}\left(\frac{9}{17}\right) = 58°$

Q. A curve is given by $x = t^2 - 1$, $y = 4t - 3$, $z = 2t^2 - 6t$. Find the angle between tangent to the curve at $t = 1$ and $t = 2$. $\left(\cos^{-1}\left(\frac{5}{3\sqrt{6}}\right)\right)$